CSE 1400 Applied Discrete Mathematics
Sets

Department of Computer Sciences

College of Engineering

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Abstract

Finite and countable sets are fundamental primitives of discrete mathematics. Operations can be defined on sets creating an "algebra." Counting the number of elements in a set and counting subsets with a certain property are fundamental in computing probabilities and statistics. Partitioning a set describes equivalences among its elementss.

Set Basics

A SET IS AN UNORDERED COLLECTION OF THINGS . The THINGS in a SET $\mathbb{A}$ are said to be elements or members of $\mathbb{A}$. The natural

If a is an element in $\mathbb{A}$ write $a \in \mathbb{A}$. Of course, it can occur that a particular element a is not a member of set $\mathbb{A}$, in which case write $a \notin \mathbb{A}$.

number 7 is a member of the set $\mathbb{O}$ of octal numerals

$$7 \in \mathbb{O} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

The natural number 8 is not a member of the set $\mathbb{O}$ of octal numerals

$$8 \notin \mathbb{O} = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

$\mathbb{A}$ can be described by listing its members as a comma separated list enclosed in curly braces $\{\}$. No element is duplicated in a set: A THING in a set is listed once and only once. $\mathbb{A}$ can be also described by comprehension. For instance,

$$\mathbb{B} = \{b : b \text{ is a bit}\}$$

For instance, $\mathbb{B} = \{0, 1\}$, is the set of bits.

More generally, $\mathbb{A}$ can be comprehended by a description

$$\mathbb{A} = \{a : \mathrm{P}(a) \text{ is } \texttt{True}\}$$

where $\mathrm{P}(a)$ is a proposition about variable a. Even more generally, $\mathbb{A}$ can be comprehended by a description

$$\mathbb{A} = \{f(a) : \mathrm{P}(a) \text{ is } \texttt{True}\}$$

where f is a function and $\mathrm{P}(a)$ is a proposition about the variable a.

When context demands it, the set of all possible things, called the UNIVERSAL SET and denoted $\mathbb{U}$, can be named. Sets can be represented by diagrams, for instance, a single set $\mathbb{X}$ is drawn as a circle inside a rectangle.

For instance, the set of even natural numbers is comprehended by the description

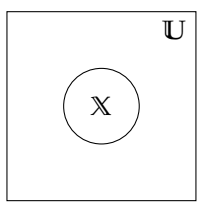
$$2\mathbb{N} = \{a : a = 2n \wedge n \in \mathbb{N}\}$$

For instance, the set of even natural numbers is comprehended by the description

$$2\mathbb{N} = \{2a : a \in \mathbb{N}\}$$

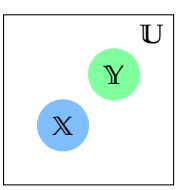
In computing practice, set comprehension requires the proposition $\mathrm{P}(a)$ to be computable, that is, there must be an algorithm that returns `True` when $a \in \mathbb{A}$ and `False` when $a \notin \mathbb{A}$.

For instance, the set of natural numbers is the universal set for many computing problems. Strings over an alphabet could be $\mathbb{U}$ in other applications.

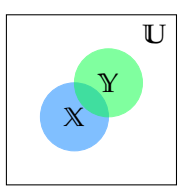


Two sets $\mathbb{X}$ and $\mathbb{Y}$ can be drawn in several relationships, called Euler diagrams.

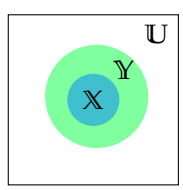
For instance, when no members of $\mathbb{X}$ are in $\mathbb{Y}$ the sets are DISJOINT. Contrapositively, no members of $\mathbb{Y}$ are in $\mathbb{X}$. Similarly, when some in $\mathbb{X}$ are in $\mathbb{Y}$ then some members of $\mathbb{Y}$ are in $\mathbb{X}$ and the sets INTERSECT. Lastly, when all members of $\mathbb{X}$ are in $\mathbb{Y}$, $\mathbb{X}$ is said to be a SUBSET of $\mathbb{Y}$.



$\mathbb{X}$ and $\mathbb{Y}$ are DISJOINT.



$\mathbb{X}$ and $\mathbb{Y}$ INTERSECT.



$\mathbb{X}$ is a SUBSET of $\mathbb{Y}$.

Characteristic functions describe these three scenarios.

Definition 1 (Characteristic Function)**.** The characteristic function of $\mathbb{A}$ is denoted $\chi_{\mathbb{A}}$ and computed by the conditional statement

$$\chi_{\mathbb{A}}(a) = \begin{cases} \texttt{False} & \text{if } a \notin \mathbb{A} \\ \texttt{True} & \text{if } a \in \mathbb{A} \end{cases}$$

When $\mathbb{X}$ and $\mathbb{Y}$ are disjoint, the values $\chi_{\mathbb{X}}(a)$ and $\chi_{\mathbb{Y}}(a)$ cannot both be `True` simultaneously. That is,

$$(\forall a)(\chi_{\mathbb{X}}(a) \wedge \chi_{\mathbb{Y}}(a) = \texttt{False})$$

When $\mathbb{X}$ and $\mathbb{Y}$ intersect, their is some element a where $\chi_{\mathbb{X}}(a)$ and $\chi_{\mathbb{Y}}(a)$ are simultaneously `True`. That is,

$$(\exists a)(\chi_{\mathbb{X}}(a) \wedge \chi_{\mathbb{Y}}(a) = \texttt{True})$$

And when $\mathbb{X}$ is a subset of $\mathbb{Y}$, every element in $\mathbb{X}$ also belongs to $\mathbb{Y}$. That is,

$$\chi_{\mathbb{X}}(a) \to \chi_{\mathbb{Y}}(a)$$

Common Sets

In mathematics, the name of a set is usually written in a font called blackboard bold. Thus, for instance, we have

- The bits or Boolean values
$$\mathbb{B} = \{0, 1\} = \{1, 0\}$$
- The digits
$$\mathbb{D} = \{0, 1, \ldots, 9\}$$
- The hexadecimal digits
$$\mathbb{H} = \{0, 1, \ldots, 9, A, \ldots, F\}$$
- The natural numbers
$$\mathbb{N} = \{0, 1, 2, \ldots\}$$
- The integers
$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$$
- The integers mod n
$$\mathbb{Z}_n = \{0, 1, 2, \ldots, (n-1)\}$$
- The rational numbers
$$\mathbb{Q} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$$
- The English alphabet
$$\mathbb{A} = \{a, b, c, \ldots, x, y, z\}$$
- The Unicode character set
$$\mathbb{U} = \{c : 0 \leq c \leq (10FFFF)_{16}\}$$

Operations On Sets

SETS CAN BE COMBINED IN SIMPLE WAYS TO CREATE COMPLEX EXPRESSIONS. Let $\mathbb{X}$ and $\mathbb{Y}$ name two sets. The UNION operator

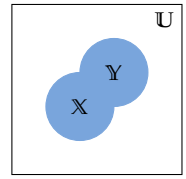
It could be that $\mathbb{X} = \mathbb{Y}$, so that two names refer to the same value.

$\mathbb{X} \cup \mathbb{Y}$ returns the set of all elements in either set: $\mathbb{X}$ or $\mathbb{Y}$.

$$\mathbb{X} \cup \mathbb{Y} = \{z : z \text{ is in } \mathbb{X} \text{ OR } z \text{ is in } \mathbb{Y}\}$$

The INTERSECTION operator $\mathbb{X} \cap \mathbb{Y}$ returns only the set of elements that are in both sets $\mathbb{X}$ and $\mathbb{Y}$.

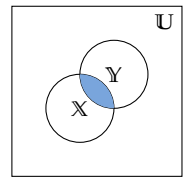
$$\mathbb{X} \cap \mathbb{Y} = \{z : z \text{ is in } \mathbb{X} \text{ AND } z \text{ is in } \mathbb{Y}\}$$



$\mathbb{X} \cup \mathbb{Y}$ is set UNION.

SET COMPLEMENT, $\overline{\mathbb{X}}$, operates on a single set and returns the elements not in $\mathbb{X}$.

$$\overline{\mathbb{X}} = \{z : z \text{ is NOT in } \mathbb{X}\}$$



$\mathbb{X} \cap \mathbb{Y}$ is the INTERSECTION operator.

For instance, let

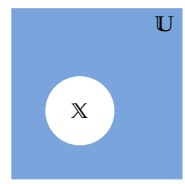
$$\mathbb{X} = \{1, 5, 9\}, \qquad \mathbb{Y} = \{2, 3, 5, 7\}, \quad \text{and} \quad \mathbb{V} = \{4, 6, 8, 9\}$$

be three subsets of the universe of decimal digits.

$$\mathbb{D} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Then

$$\begin{aligned} \mathbb{X} \cap \mathbb{V} &= \{9\} \\ \overline{X} &= \{0, 2, 3, 4, 6, 7, 8\} \\ \mathbb{X} \cup \mathbb{Y} &= \{1, 2, 3, 5, 7, 9\} \\ (\mathbb{X} \cap \mathbb{Y}) \cup \overline{\mathbb{V}} &= \{0,1,2,3,5,7\} \end{aligned}$$



$\overline{\mathbb{X}}$ is the SET COMPLEMENT operator.

Precedence of Set Operations

Precedence and associativity determine the ORDER in which operations are performed. For sets, complement is computed before intersection, which is before union. All operations are computed left-to-right unless parenthesis or other brackets are used to specify order.

In arithmetic you learn that exponentiation is performed before multiplication, which are performed before addition. Furthermore, the associativity of addition and multiplication is left-to-right, but exponentiation is right-to-left. For instance,

$$6 + 5 \cdot 2^{2^3} = 6 + 5 \cdot 2^8 = 6 + 5 \cdot 256$$

For instance, when $\mathbb{A} = \{0\}$, $\mathbb{B} = \{0, 1\}$ and $\mathbb{C} = \{1, 2\}$ compute

$$\begin{aligned} \mathbb{A} \cup \mathbb{B} \cap C &= \{0\} \cup \{1\} \\ &= \{0, 1\} \\ (\mathbb{A} \cup \mathbb{B}) \cap C &= \{0, 1\} \cap \{1, 2\} \\ &= \{1\} \end{aligned}$$

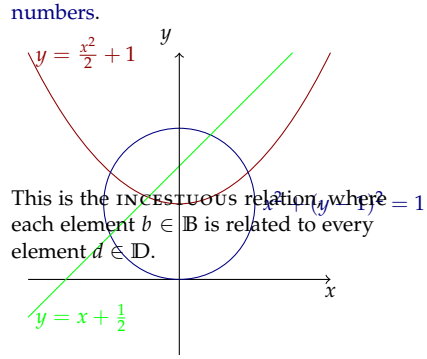
Cartesian Products

CARTESIAN PRODUCTS PROVIDES THE FOUNDATION FOR BUILDING RELATIONS AND FUNCTIONS.

The Cartesian product of $\mathbb{X}$ and $\mathbb{Y}$ is the set of ordered (x, y) that relates every value $x \in \mathbb{X}$ with every value $y \in \mathbb{Y}$. The Cartesian product of $\mathbb{X}$ and $\mathbb{Y}$ is written

$$\mathbb{X} \times \mathbb{Y} = \{(x, y) : x \in \mathbb{X} \wedge y \in \mathbb{Y}\}$$

College algebra teaches how to draw functions and relations on the real numbers.



This is the INCESTUOUS relation, where each element $b \in \mathbb{B}$ is related to every element $d \in \mathbb{D}$.

The Cartesian product $\mathbb{X} \times \mathbb{Y}$ is TWO DIMENSIONAL: It can be represented as a set of ordered pairs, for instance,

$$\mathbb{B} \times \mathbb{D} = \{(0, 0), (0, 1), \ldots, (0, 9), (1, 0), (1, 1), \ldots, (1, 9)\}$$

The Cartesian product can be represented as a table, for instance $\mathbb{B} \times \mathbb{D}$ is the table where each entry is `True`, represented by 1.

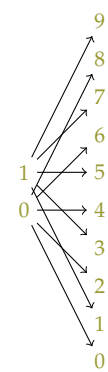
$\mathbb{B} \times \mathbb{D}$										
	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1

Subsets of $\mathbb{B} \times \mathbb{D}$ are called relations from $\mathbb{B}$ to $\mathbb{D}$. For instance, the the even-odd relation can be represented by the table where row 0 picks out the even digits as `True` row 1 picks out the odd digits as `True`

Even-Odd Relation										
	0	1	2	3	4	5	6	7	8	9
0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	0	1

A Cartesian product can be represented as a NODE-EDGE GRAPH. Sub-graphs of a complete graph arise in computing practice.

For instance, the even-odd relation



Subset of a Set

OFTEN IT IS USEFUL TO TALK ABOUT A COLLECTION OF SOME ELEMENTS, BUT PERHAPS NOT ALL ELEMENTS, OF A SET. Such a collection is called a SUBSET. The set of composite digits $\mathbb{C} = \{4, 6, 8, 9\}$ is a subset of the digits $\mathbb{D}$.

The subset relation between two sets is very much like the less than relation between two integers. A proper subset is strictly smaller than its super-set, just as 5 is strictly less than 7.

$$\mathbb{C} = \{4, 6, 8, 9\} \subset \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = \mathbb{D}$$

To allow for the possibility of equality, write $a \leq b$ for integers and $\mathbb{A} \subseteq \mathbb{X}$ for sets.

If there is some element in $\mathbb{A}$ that is not in $\mathbb{X}$, then $\mathbb{A}$ is not a subset of $\mathbb{X}$. Stated contra-positively, if $\mathbb{A} \subseteq \mathbb{X}$, then every member of $\mathbb{A}$ is a member of $\mathbb{X}$.

If $\mathbb{A}$ is not a subset of $\mathbb{X}$, write $\mathbb{A} \not\subseteq \mathbb{X}$

There is one and only one smallest set. It is the set without any elements, and it is called the EMPTY SET. The EMPTY SET is denoted by the symbol $\emptyset$. Interestingly, the EMPTY SET is a subset of any set $\mathbb{A}$. If it were not, there would have to be an element in $\emptyset$ that is not in $\mathbb{A}$, but there can be no such element a since $\emptyset$ contains no elements. The vacuous proof that $\emptyset \subseteq \mathbb{A}$ is that every element in $\emptyset$ is an element in $\mathbb{A}$.

Cardinality of a Set

INFORMALLY, THE CARDINALITY OF A SET $\mathbb{A}$ IS THE COUNT OF ELEMENTS IN $\mathbb{A}$. If this count can be NAMED by a natural number, then $\mathbb{A}$ is a finite set. If $\mathbb{A}$ is not a finite set, it is an INFINITE set. There are at least two types of infinite sets. Those that are COUNTABLE and those that are not. The natural numbers is the definition of COUNTABLE, INFINITE set. Any other set $\mathbb{A}$ is countable if there is a one-to-one and onto function f mapping $\mathbb{N}$ to $\mathbb{A}$. For instance, the set of fractions

Set $\mathbb{A}$ is finite with cardinality n when the elements of $\mathbb{A}$ can be put into a one-to-one and onto correspondence with the set $\{0, 1, \ldots, n-1\}$.

A finite set is also countable.

The idea behind COUNTABILITY is that given *any* natural number n we can (eventually) count to it.

It is not difficult to imagine examples of countable sets. There are examples that are counter-intuitive.

$$\mathbb{A} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

is countable because the function

$$f(n) = \frac{1}{n+1}$$

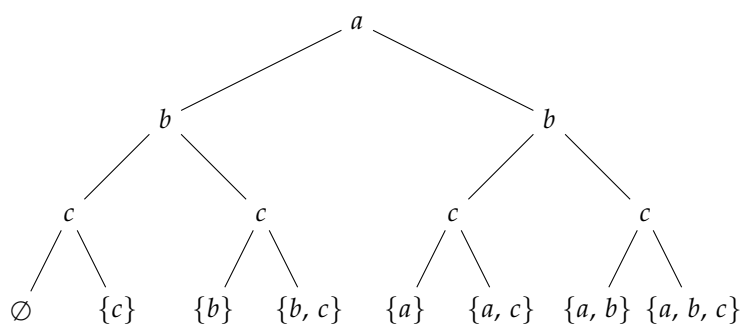
establishes a one-to-one and onto correspondence between the two sets.

Power Set of a Set

THE COLLECTION OF ALL SUBSETS OF A SET $\mathbb{A}$ IS CALLED $\mathbb{A}$'S POWER SET. If $\mathbb{A}$ has 3 elements then $\mathbb{A}$'s power set contains $2^3 = 8$ elements, each of which is a subset of $\mathbb{A}$.

The *power set* of $\mathbb{A}$ is denoted by $2^{\mathbb{A}}$.

Think of constructing a subset. For each element there are two choices: include it in the subset or leave it out. Consider the decision tree for constructing subsets of $\mathbb{A} = \{a, b, c\}$. When the left branch is followed, the element is not included in the subset. When the right branch is followed, the element is included in the subset.



Theorem 1 (Power Set Cardinality)**.** *A set with n elements has 2^n subsets, that is, the power set of $\mathbb{A}$ contains all 2^n subsets of $\mathbb{A}$ when $|\mathbb{A}| = n$.*

If $|\mathbb{A}| = n$, then $|2^{\mathbb{A}}| = 2^{|\mathbb{A}|} = 2^n$.

Binomial Coefficients

GIVEN A SET $\mathbb{A}$ WITH CARDINALITY n, HOW MANY m-ELEMENT SUBSETS DOES $\mathbb{A}$ HAVE? The binomial coefficient $\binom{n}{m}$ denotes this number. When a step of an algorithm requires the selection of a subset of values, the chosen subset is called a combination. The size (cardinality) n of the SAMPLE SPACE and the size (cardinality) m of the COMBINATION determine the count of combinations that can be constructed. The example above shows that a three element set has

The symbol $\binom{n}{m}$ is called "n choose m".

- 1 subset without any elements, $\binom{3}{0} = 1$
- 3 subsets with one element, $\binom{3}{1} = 3$
- 3 subsets with two elements, $\binom{3}{2} = 3$
- 1 subset with three elements, $\binom{3}{3} = 1$

Any given set $\mathbb{A}$ has 1 subset without any elements, the empty . Also, there is 1 subset of $\mathbb{A}$ that contains every element in $\mathbb{A}$, the set $\mathbb{A}$ itself.

There is only one way to choose nothing at all.

There is only one way to choose everything.

The BOUNDARY CONDITIONS on the binomial coefficients are $\binom{n}{0} = \binom{n}{n} = 1$.

It is convenient to declare that the empty set $\emptyset$ is a subset of any given set $\mathbb{A}$.

Describing how to construct something often leads to a method for counting those things.

For small sets each subset can be listed. Let $\mathbb{A}$ be a set with cardinality $|\mathbb{A}| = n$. There are 2^n subsets of $\mathbb{A}$. Figure 1 shows the $2^3 = 8$ subsets of $\mathbb{A} = \{0, 1, 2\}$.

The subsets of $\{0, 1, 2, 3\}$ can be constructed recursively in terms of the subsets of $\{0, 1, 2\}$. For instance, to build all 2-element subsets of $\mathbb{N}_3 = \{0, 1, 2, 3\}$ use the one and two-element subsets of $\mathbb{N}_2 = \{0, 1, 2\}$. There are two cases to consider.

1. Element 3 is in a 2-element subset of $\mathbb{N}_3$.
2. Element 3 is not in a 2-element subset of $\mathbb{N}_3$.

These two cases are handled by steps below that build a 2-element subset of $\mathbb{N}_3$.

1. Take the union of $\{3\}$ with each 1-element subset of $\{0, 1, 2\}$.
2. Include in the collection of subsets each already built 2-element subset of $\mathbb{N}_2 = \{0, 1, 2\}$.

Using the binomial coefficient $\binom{n}{m}$ to name the count of 2-elements subsets of a 4-element set, write

$$\binom{4}{2} = \binom{3}{1} + \binom{3}{2} = 3 + 3 = 6$$

These six subsets are shown in figure 2.

Counting Bit Strings

THERE ARE 2^n DIFFERENT BITS STRINGS OF LENGTH n. The number of bit strings of length n with m 1's and $n - m$ 0's is equal to the number of m-element subsets of an n-element set. For instance, there are 6 bits strings of length 4 with two bits set to 1 and two bits set to 0. Recursion can be used to construct an n-long bit string with m 1's. There are two cases.

Think of the bit string as a record that elements are in the subset (the bit is 1) or not in the subset (the bit is 0.)
$\{0011, 0101, 0110, 1001, 1010, 1100\}$

1. Let s_{n-1} be an $n-1$-long bit string with $m-1$ 1's.
2. Let t_{n-1} be an $n-1$-long bit string with m 1's.

Then

$1s_{n-1}$ is an bit string of length n with m 1's.

and

$0t_{n-1}$ is an bit string of length n with m 1's.

The strings $1s_{n-1}$ and $0t_{n-1}$ are different Let $\binom{n-1}{m-1}$ denote the number of strings like s_{n-1} and let $\binom{n-1}{m}$ denote the number of strings like t_{n-1}. Then the number of n-long bit strings with m 1's can be computed by the recursion equation

$1s_{n-1}$ has a leading 1 while $0t_{n-1}$ has a leading 0.

This recursion equation is known as Pascal's identity.

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$$

Table 1 lists the binomial coefficients "n choose m" for pairs of digits.

Notice how the sum of two values in one row determine the value of a term in the next row.

$21 + 35 = 56$ $\qquad \binom{7}{2} + \binom{7}{3} = \binom{8}{3}$

$15 + 6 = 21$ $\qquad \binom{6}{5} + \binom{6}{5} = \binom{7}{5}$

$$\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$$

Boolean Laws

The power set $2^{\mathbb{U}}$ of a set $\mathbb{U}$ together with the union, intersection, and set complement operations form a Boolean algebra. That is, the following properties hold for subsets $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ of $\mathbb{U}$.

Boolean logic on PROPOSITIONS using OR ($\vee$), AND ($\wedge$), and NOT ($\neg$) operations form a similar Boolean algebra.

Identity Laws

Subsets			
$\emptyset$	$\{0\}$	$\{0, 1\}$	$\{0, 1, 2\}$
	$\{1\}$	$\{0, 2\}$	
	$\{2\}$	$\{1, 2\}$	

Figure 1: The $2^3 = 8$ subsets of $\{0, 1, 2\}$.

Subsets				
$\emptyset$	$\{0\}$	$\{0, 3\}$	$\{0, 1, 3\}$	$\{0, 1, 2, 3\}$
	$\{1\}$	$\{1, 3\}$	$\{0, 2, 3\}$	
	$\{2\}$	$\{2, 3\}$	$\{1, 2, 3\}$	
	$\{3\}$	$\{0, 1\}$	$\{0, 1, 2\}$	
		$\{0, 2\}$		
		$\{1, 2\}$		

Figure 2: The 2^4 subsets of $\{0, 1, 2, 3\}$.

Table 1: Binomial coefficients $\binom{n}{m}$ count the number of different ways to choose m elements from a set of n elements.

	Binomial Coefficients $\binom{n}{m}$ Choose m									
		0	1	2	3	4	5	6	7	8	9
	0	1									
	1	1	1								
	2	1	2	1							
	3	1	3	3	1						
n	4	1	4	6	4	1					
	5	1	5	10	10	5	1				
	6	1	6	15	20	15	6	1			
	7	1	7	21	35	35	21	7	1		
	8	1	8	28	56	70	56	28	8	1	
	9	1	9	36	84	126	126	84	36	9	1

1. $\mathbb{X} \cup \emptyset = \mathbb{X}$
2. $\mathbb{X} \cap \mathbb{U} = \mathbb{X}$

Complement Laws

1. $\mathbb{X} \cup \overline{\mathbb{X}} = \mathbb{U}$
2. $\mathbb{X} \cap \overline{\mathbb{X}} = \emptyset$

Associative Laws

1. $(\mathbb{X} \cup \mathbb{Y}) \cup \mathbb{Z} = \mathbb{X} \cup (\mathbb{Y} \cup \mathbb{Z})$
2. $(\mathbb{X} \cap \mathbb{Y}) \cap \mathbb{Z} = \mathbb{X} \cap (\mathbb{Y} \cap \mathbb{Z})$

Commutative Laws

1. $\mathbb{X} \cup \mathbb{Y} = \mathbb{Y} \cup \mathbb{X}$
2. $\mathbb{X} \cap \mathbb{Y} = \mathbb{Y} \cap \mathbb{X}$

Distributive Laws

1. $\mathbb{X} \cup (\mathbb{Y} \cap \mathbb{Z}) = (\mathbb{X} \cup \mathbb{Y}) \cap (\mathbb{X} \cup \mathbb{Z})$
2. $\mathbb{X} \cap (\mathbb{Y} \cup \mathbb{Z}) = (\mathbb{X} \cap \mathbb{Y}) \cup (\mathbb{X} \cap \mathbb{Z})$

Many theorems can be derived from these fundamental laws, for instance De Morgan's laws

$$\overline{\mathbb{X} \cup \mathbb{Y}} = \overline{\mathbb{X}} \cap \overline{\mathbb{Y}}$$
$$\overline{\mathbb{X} \cap \mathbb{Y}} = \overline{\mathbb{X}} \cup \overline{\mathbb{Y}}$$

In English, NOT (x OR y) IS NOT x AND NOT y, and NOT x AND y IS NOT x OR NOT y.

A proof of theorems such as De Morgan's laws can be given by examining all cases. The case examination can be organized into a truth table.

Cases		Results	
$z \in \mathbb{X}$	$z \in \mathbb{Y}$	$z \in \overline{\mathbb{X} \cup \mathbb{Y}}$	$z \in \overline{\mathbb{X}} \cap \overline{\mathbb{Y}}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

Cases		Results	
$z \in \mathbb{X}$	$z \in \mathbb{Y}$	$z \in \overline{\mathbb{X} \cap \mathbb{Y}}$	$z \in \overline{\mathbb{X}} \cup \overline{\mathbb{Y}}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Partition of a Set

To PARTITION IS TO SEPARATE INTO PARTS. For instance the even-odd relation partitions the digits into two subsets.

$$\{0, 2, 4, 6, 8\} \quad \text{and} \quad \{1, 3, 5, 7, 9\}$$

A similar instance is the mod 3 relation which partitions the digits into three subsets based on their remainder upon division by 3.

$$\{0, 3, 6, 9\}, \quad \{1, 4, 7\}, \quad \text{and} \quad \{2, 5, 8\}$$

For sets with small cardinality it is possible to list all of the parti-

tions.

		Partitions with	
Set	1 subset	2 subset	3 subset
$\{0\}$	$\{\{0\}\}$		
$\{0, 1\}$	$\{\{0, 1\}\}$	$\{\{0\}, \{1\}\}$	
$\{0, 1, 2\}$	$\{\{0, 1, 2\}\}$	$\{\{0\}, \{1, 2\}\}$ $\{\{1\}, \{0, 2\}\}$ $\{\{2\}, \{0, 1\}\}$	$\{\{0\}, \{1\}, \{2\}\}$
$\{0, 1, 2, 3\}$	$\{\{0, 1, 2, 3\}\}$	$\{\{0, 1, 2\}, \{3\}\}$ $\{\{0, 3\}, \{1, 2\}\}$ $\{\{0\}, \{1, 2, 3\}\}$ $\{\{1, 3\}, \{0, 2\}\}$ $\{\{1\}, \{0, 2, 3\}\}$ $\{\{2, 3\}, \{0, 1\}\}$ $\{\{2\}, \{0, 1, 3\}\}$	$\{\{0\}, \{1, 2\}, \{3\}\}$ $\{\{1\}, \{0, 2\}, \{3\}\}$ $\{\{2\}, \{0, 1\}, \{3\}\}$ $\{\{0\}, \{1\}, \{2, 3\}\}$ $\{\{0\}, \{1, 3\}, \{2\}\}$ $\{\{0, 3\}, \{1\}, \{2\}\}$

The number ways an n-element set can be partitioned into m subsets is a Stirling number of the second kind denoted by $\left\{ {n \atop m} \right\}$. A Stirling number of the second kind can be computed by the recurrence equation

$$\left\{ {n \atop m} \right\} = \left\{ {n-1 \atop m-1} \right\} + m \left\{ {n-1 \atop m} \right\}$$

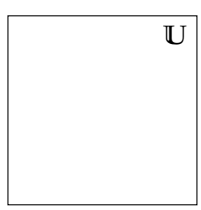
with boundary conditions

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = 0 \quad \text{and} \quad \left\{ {n \atop n} \right\} = 1, \text{ for } n > 0$$

These ideas occur in the study of equivalence relations, which are relations that partition sets into equivalence classes.

Venn and Euler Diagrams

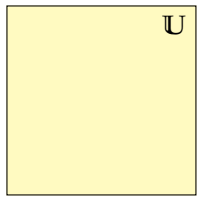
Venn's diagramming technique show relationships between sets. When used in logic, Venn diagrams show relationships among propositions. Venn diagramming begins with an rectangle representing the universe of elements. If the rectangle is not shaded it represents the empty set $\emptyset$.



The empty set $\emptyset$

Mathematicians shade the non-empty region. Logician shade the empty region. Students suffer.

Shading a region in a Venn diagram indicates it is not empty; it is full.

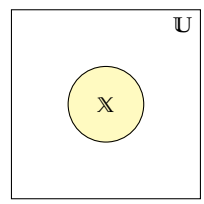


A non-empty universe

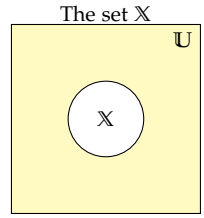
Using this shading rule two REGIONS that can be identified: The universe $\mathbb{U}$ and the empty set $\emptyset$.

When one subset, call it $\mathbb{X}$, is drawn as a circle inside the universe, two additional REGIONS that can recognized: The set $\mathbb{X}$ and its complement $\overline{\mathbb{X}}$. When two intersecting subsets, call them $\mathbb{X}$ and $\mathbb{Y}$, are drawn as circles inside the universe, an additional 12 REGIONS that can identified. One way to understand this is to name the four regions in the Venn diagram. Call them a, b, c and d, and note they can be expressed as intersections of $\mathbb{X}$ and $\mathbb{Y}$ or their complements.

Region	Set Expression
a	$\overline{\mathbb{X}} \cap \overline{\mathbb{Y}}$
b	$\mathbb{X} \cap \overline{\mathbb{Y}}$
c	$\overline{\mathbb{X}} \cap \mathbb{Y}$
d	$\mathbb{X} \cap \mathbb{Y}$

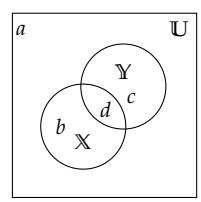


The set $\mathbb{X}$



$\overline{\mathbb{X}}$: The complement of $\mathbb{X}$

You can construct the empty set from the set of regions $\{a, b, c, d\}$ by not choosing any of them. The universe is constructed by the opposite: Choose every region and compute their union. These are the boundary conditions. There are interior cases: Choose one REGION, Choose two REGIONS, or choose three REGIONS. These cases can be diagrammed.



From the four regions $\{a, b, c, d\}$ choose

No REGIONS	One REGION	Two REGIONS	Three REGIONS	Four REGIONS

A functional relationship between the number intersecting subsets and number of different REGIONS that can be described can be expressed by a function

$$r(n) = 2^{2^n}$$

where n is the number of intersecting subsets and $r(n)$ counts the total number of distinct REGIONS that can be constructed.

Count of Subsets	0	1	2	3	4
Count of Regions	2	4	16	256	65536
Function	$\lvert\text{Regions}\rvert = 2^{2^{\lvert\text{Subsets}\rvert}}$				

Problems on Sets

1. Describe the following sets. List a representative sample of elements establishing a pattern. Give a function that computes the elements.
   (a) The set of even integers.
   (b) The set of odd integers.
   (c) The set of integers that have a remainder of 2 when divided by 3.
   (d) The Mersenne numbers.
   (e) The triangular numbers.
   (f) The set of solutions to the polynomial equation $x^2 - x - 1 = 0$.
   (g) The set of solutions to the equation $|x| - 1 = 0$.
2. What notation would be used to stand for the following phrases?
   (a) x is an element of $\mathbb{X}$.
   (b) x is not an element of $\mathbb{X}$.
   (c) $\mathbb{X}$ is a subset of $\mathbb{Y}$.
   (d) $\mathbb{X}$ is a proper subset of $\mathbb{Y}$.
   (e) $\mathbb{X}$ is not a subset of $\mathbb{Y}$.
   (f) The union of $\mathbb{X}$ and $\mathbb{Y}$.
   (g) The intersection of $\mathbb{X}$ and $\mathbb{Y}$.
   (h) The complement of $\mathbb{X}$.
   (i) The empty set
3. Answer `True` or `False`.
   (a) $2 \in \{\{2\}\}$.
   (b) $2 \in \{2\}$.
   (c) $2 \notin \{2\}$.
   (d) $\emptyset \in \{\emptyset\}$.
   (e) $0 \in \emptyset$.
   (f) $\emptyset \subseteq \{x\}$.
   (g) $\emptyset = \{\emptyset\}$.
   (h) $\{x\} \subseteq \{x\}$.
   (i) $\{x\} \in \{x\}$.
4. Below are standard names for sets that occur in computing. Describe these sets.
   (a) $\mathbb{B}$
   (b) $\mathbb{H}$
   (c) $\mathbb{N}$
   (d) $\mathbb{Z}$
   (e) $\mathbb{Z}^+$
   (f) $\mathbb{Q}$
   (g) $\mathbb{R}$
   (h) $\emptyset$
5. What is the cardinality of each of these sets?
   (a) $\emptyset$.
   (b) $\{\emptyset\}$.
   (c) $\mathbb{D}$.
   (d) $\{x : x^2 - x - 1 = 0\}$.
   (e) $\mathbb{N}$.
   (f) $\mathbb{Q}$.
   (g) $\mathbb{R}$.

6. What is the power set of each of these sets?

   (a) $\{0\}$.
   (b) $\{x : x^2 - x - 1 = 0\}$.
   (c) $\emptyset$.
   (d) $\{\emptyset\}$.

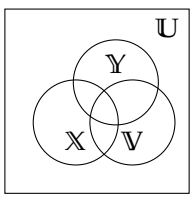
7. What is the cardinality of each of these sets?

   (a) The power set of $\{0\}$.
   (b) The power set of $\{x : x^2 - x - 1 = 0\}$.
   (c) The power set of $\mathbb{D}$, the set of digits.
   (d) The power set of $\mathbb{H}$, the set of hexadecimal numerals.

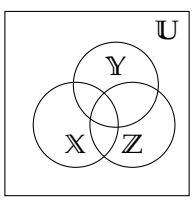
8. Describe how to define the SET DIFFERENCE operator: $\mathbb{Y} - \mathbb{X}$ using the standard union, intersection, and set complement operators.

9. Shade the Venn diagram to indicate the given region is not empty.

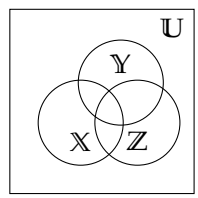
   (a) $\mathbb{X} \cap \overline{\mathbb{Y}}$.



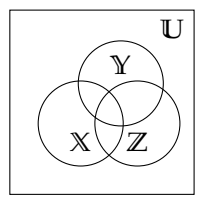
   (b) $\mathbb{X} \cap \mathbb{Y} \cap \mathbb{V}$.



   (c) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{Z}$.

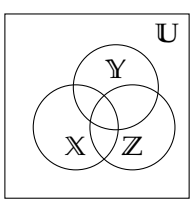


   (d) $\overline{\mathbb{X}} \cap \overline{\mathbb{Y}} \cup \overline{\mathbb{Z}}$.

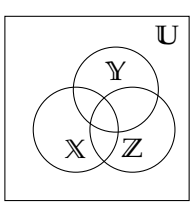


10. Shade the Venn diagram to indicate the given region is NOT empty.

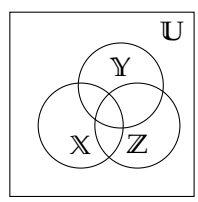
   (a) $\mathbb{X} \cap \mathbb{Y}$.



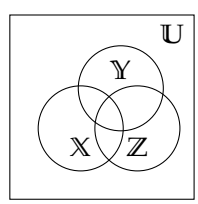
   (b) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{Z}$.



   (c) $\overline{\mathbb{X}} \cap \mathbb{Y} \cap \mathbb{Z}$.



   (d) $\overline{\mathbb{X}} \cap \overline{\mathbb{Y}} \cup \overline{\mathbb{Z}}$.



11. Let $\mathbb{X}$ and $\mathbb{Y}$ be subsets of some universal set $\mathbb{U}$. Are the following statements `True` or `False`?

(a) $\overline{\overline{X}} = \overline{X}$
(b) $\mathbb{X} \cup \emptyset = \mathbb{X}$
(c) $\mathbb{X} \cap \mathbb{U} = \mathbb{X}$
(d) $\mathbb{X} \cap \overline{\mathbb{U}} = \mathbb{U}$
(e) $\mathbb{X} \cap \overline{\mathbb{U}} = \emptyset$
(f) $\overline{\mathbb{X} \cap \mathbb{Y}} = \overline{\mathbb{X}} \cup \overline{\mathbb{Y}}$
(g) $\mathbb{X} \cap \mathbb{Y} \subseteq \mathbb{X}$

12. Let $\mathbb{M}_{10} = \{0, 1, 3, 7, 15, 31, 63, 127, 255, 511\}$ be the set of the first 10 Mersenne numbers. Let $\mathbb{T}_{10} = \{0, 1, 3, 6, 10, 15, 21, 28, 36, 45\}$ be the set of the first 10 triangular numbers. Find

(a) $\mathbb{M}_{10} \cup \mathbb{T}_{10}$.
(b) $\mathbb{M}_{10} \cap \mathbb{T}_{10}$.
(c) $\mathbb{M}_{10} - \mathbb{T}_{10}$.
(d) $\mathbb{T}_{10} - \mathbb{M}_{10}$.

13. Using the three sets

$$\mathbb{X} = \{1,2,3,4,5\} \qquad \mathbb{Y} = \{0,2,4,6,8\} \qquad \mathbb{Z} = \{0,3,5,9,\}$$

over the universe of digits

$$\mathbb{D} = \{0,1,2,3,4,5,6,7,8,9\}$$

compute the following set operations

(a) $\mathbb{X} \cup \mathbb{Y}$
(b) $\mathbb{X} \cup \overline{\mathbb{Z}}$
(c) $(\mathbb{X} \cup \mathbb{Y}) \cap \mathbb{Z}$
(d) $\overline{(\mathbb{X} \cup \mathbb{Y})} \cap \mathbb{Z}$

14. Using the three sets

$$\mathbb{X} = \{1,2,3,4,5\} \qquad \mathbb{Y} = \{0,2,4,6,8\} \qquad \mathbb{Z} = \{0,3,5,9,\}$$

over the universe of digits

$$\mathbb{D} = \{0,1,2,3,4,5,6,7,8,9\}$$

write expressions using set operators *union* $\cup$, *intersection* $\cap$ and *complement* that are equal to the following sets.

(a) $\mathbb{S} = \{6,8\}$
(b) $\mathbb{T} = \{1,2,4\}$
(c) $\mathbb{V} = \{0,1,2,3,4,5,6,8\}$
(d) $\mathbb{W} = \{3,5\}$

15. Consider the "set" $\mathbb{S} = \{x : x \notin x\}$.

(a) Show that $\mathbb{S} \in \mathbb{S}$ is a contradiction.
(b) Show that $S \notin \mathbb{S}$ is a contradiction.

This is known as Russell's paradox.

16. What is the "time" taken to compute the binomial coefficient using the factorial formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}?$$

Assume that "time" is measured by the number of multiplications in the reduced formula

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 2}$$

17. What is the "time" taken to compute the binomial coefficient using the recursive (Pascal's) formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}?$$

Assume that "time" is measured by the minimal number of additions required to build Pascal's triangle to $\binom{n}{k}$. In particular, you may assume that $k = \min\{k, n-k\}$ and use the symmetry of the triangle.