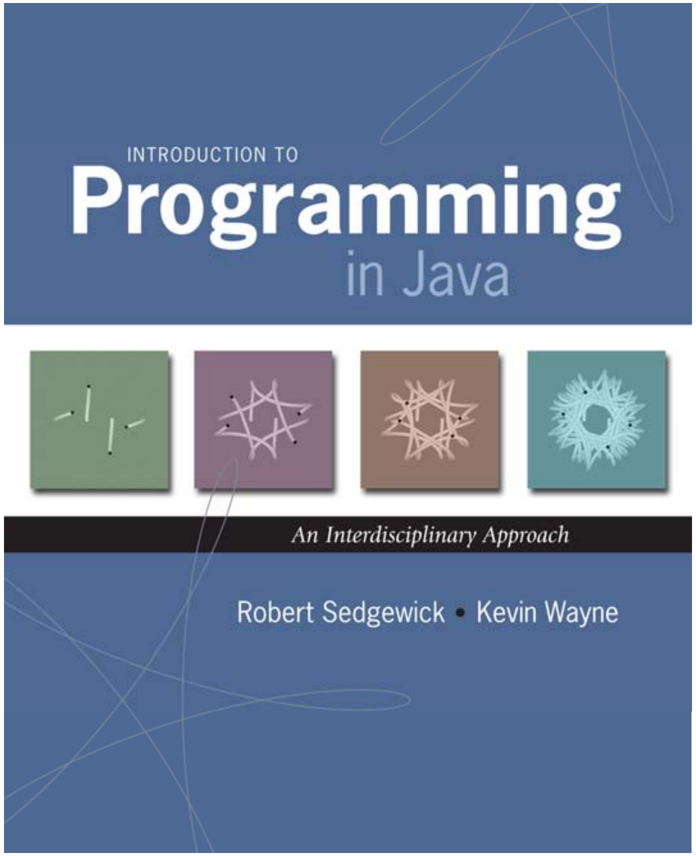
4.2 Sorting and Searching



Sequential Search

Sequential search. Scan through array, looking for key.

- Search hit: return array index.
- Search miss: return -1.

```
public static int search(String key, String[] a) {
   int N = a.length;
   for (int i = 0; i < a.length; i++)
      if (a[i].compareTo(key) == 0)
         return i;
   return -1;

}
```

Search Client: Exception Filter

Exception filter. Read a sorted list of strings from a whitelist file, then print out all strings from standard input not in the whitelist.

```
public static void main(String[] args) {
   In in = new In(args[0]);
   String s = in.readAll();
   String[] words = s.split("\\s+");
   while (!StdIn.isEmpty()) {
      String key = StdIn.readString();
      if (search(key, words) == -1)
         StdOut.println(key);
      }
   }
}
```

```
more test.txt
bob@office
carl@beach
marvin@spam
bob@office
bob@office
mallory@spam
dave@boat
eve@airport
alice@home
```

```
% more whitelist.txt
alice@home
bob@office
carl@beach
dave@boat

% java BinarySearch whitelist.txt < test.txt
marvin@spam
mallory@spam
eve@airport
```

Searching Challenge 1

Q. A credit card company needs to whitelist 10 million customer account numbers, processing 1,000 transactions per second.

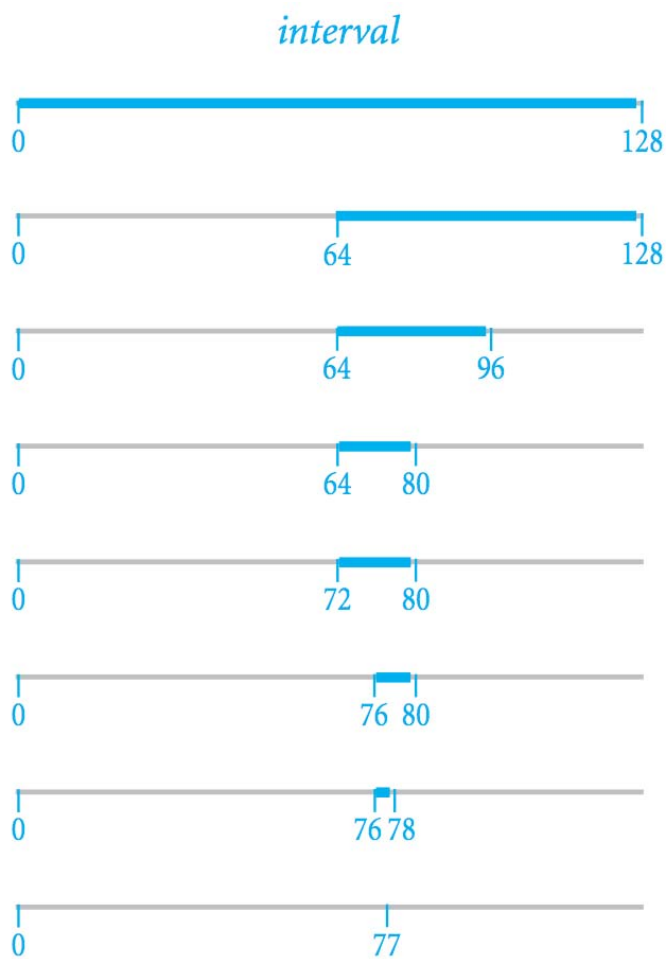
Using sequential search, what kind of computer is needed?

A. Toaster
B. Cellphone
C. Your laptop
D. Supercomputer
E. Google server farm

Binary Search

Twenty Questions

Intuition. Find a hidden integer.



size	*Q*	*A*
128	< 64 ?	*false*
64	< 96 ?	*true*
32	< 80 ?	*true*
16	< 72 ?	*false*
8	< 76 ?	*false*
4	< 78 ?	*true*
2	< 77 ?	*false*
1	= 77	

Binary Search

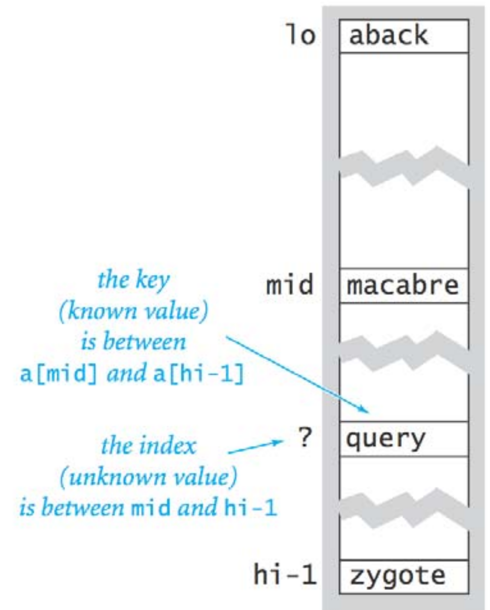
Main idea.

- Sort the array (stay tuned).
- Play "20 questions" to determine index with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, …

Binary search.

- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.



Binary search in an array (one step)

Binary Search: Java Implementation

Invariant. Algorithm maintains `a[lo]` ≤ `key` ≤ `a[hi-1]`.

```
public static int search(String key, String[] a) {
   return search(key, a, 0, a.length);
}

public static int search(String key, String[] a, int lo, int hi) {
   if (hi <= lo) return -1;
   int mid = lo + (hi - lo) / 2;
   int cmp = a[mid].compareTo(key);
   if      (cmp > 0) return search(key, a, lo, mid);
   else if (cmp < 0) return search(key, a, mid+1, hi);
   else              return mid;
}
```

Java library implementation: `Arrays.binarySearch()`

Binary Search: Mathematical Analysis

Analysis. To binary search in an array of size N: do one compare, then binary search in an array of size $N / 2$.

$$N \to N/2 \to N/4 \to N/8 \to \ldots \to 1$$

Q. How many times can you divide a number by 2 until you reach 1?

A. $\log_2 N$.

$$1$$
$$2 \to 1$$
$$4 \to 2 \to 1$$
$$8 \to 4 \to 2 \to 1$$
$$16 \to 8 \to 4 \to 2 \to 1$$
$$32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$
$$1024 \to 512 \to 256 \to 128 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$$

Searching Challenge 2

Q. A credit card company needs to whitelist 10 million customer account numbers, processing 1,000 transactions per second.

Using binary search, what kind of computer is needed?

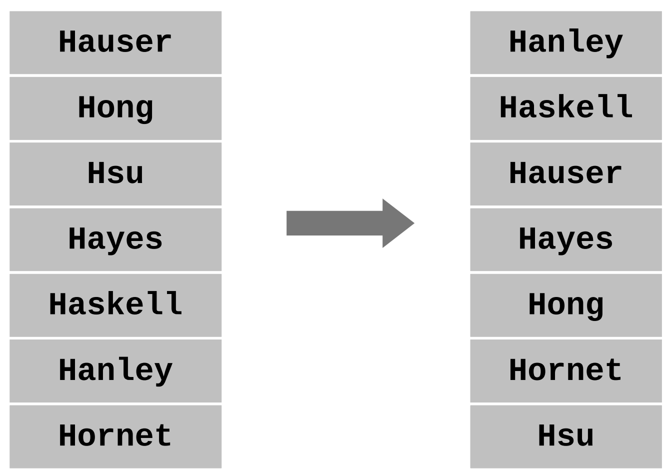
A. Toaster
B. Cell phone
C. Your laptop
D. Supercomputer
E. Google server farm

Sorting

Sorting

Sorting problem. Rearrange N items in ascending order.

Applications. Statistics, databases, data compression, bioinformatics, computer graphics, scientific computing, (too numerous to list), ...



Insertion Sort

Insertion Sort

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
6	6	and	had	him	his	was	you	the	but
6	5	and	had	him	his	was	the	you	but
6	4	and	had	him	his	the	was	you	but
		and	had	him	his	the	was	you	but

Inserting a[6] into position by exchanging with larger entries to its left

Insertion sort.

- Brute-force sorting solution.
- Move left-to-right through array.
- Exchange next element with larger elements to its left, one-by-one.

i	j	a[0]	a[1]	a[2]	a[3]	a[4]	a[5]	a[6]	a[7]
		was	had	him	and	you	his	the	but
1	0	had	was	him	and	you	his	the	but
2	1	had	him	was	and	you	his	the	but
3	0	and	had	him	was	you	his	the	but
4	4	and	had	him	was	you	his	the	but
5	3	and	had	him	his	was	you	the	but
6	4	and	had	him	his	the	was	you	but
7	1	and	but	had	him	his	the	was	you
		and	but	had	him	his	the	was	you

Inserting a[1] *through* a[N-1] *into position (insertion sort)*

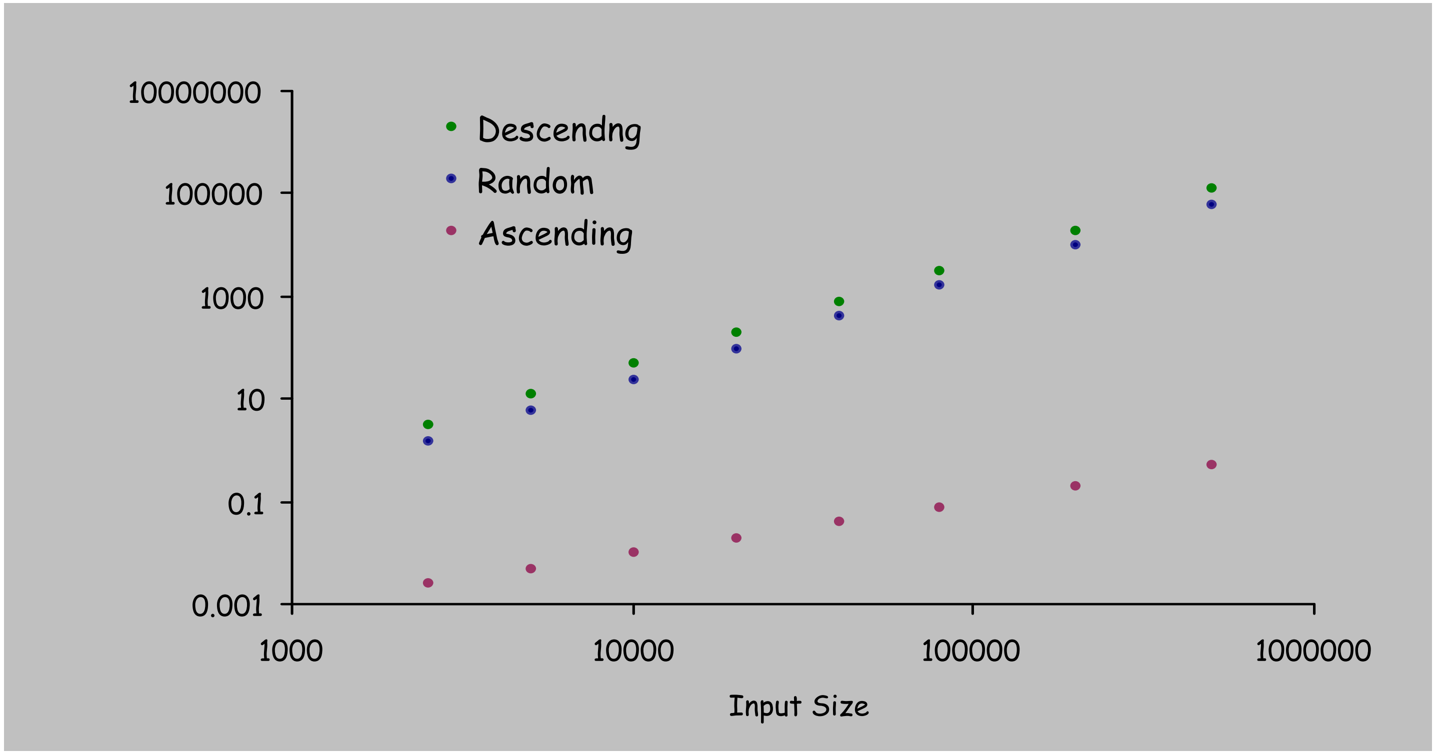
Insertion Sort: Java Implementation

```
public class Insertion {

   public static void sort(String[] a) {
      int N = a.length;
      for (int i = 1; i < N; i++)
         for (int j = i; j > 0; j--)
            if (a[j-1].compareTo(a[j]) > 0)
               exch(a, j-1, j);
            else break;
   }

   private static void exch(String[] a, int i, int j) {
      String swap = a[i];
      a[i] = a[j];
      a[j] = swap;
   }
}
```

Insertion Sort: Empirical Analysis

Observation. Number of compares depends on input family.

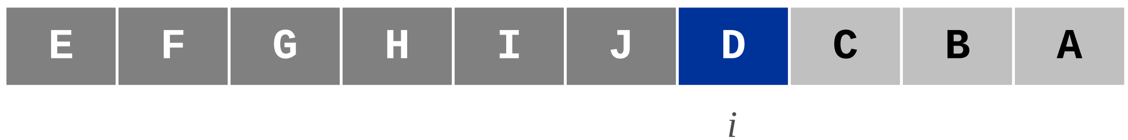
- Descending: $\sim N^2 / 2$.
- Random: $\sim N^2 / 4$.
- Ascending: $\sim N$.



Insertion Sort: Mathematical Analysis

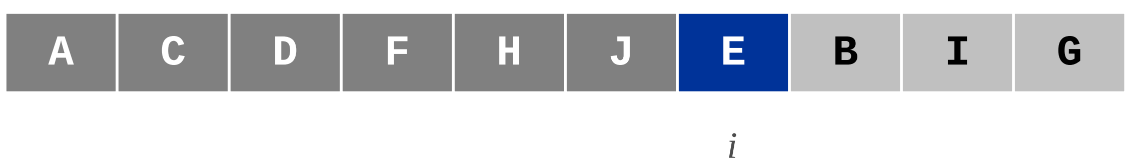
Worst case. [descending]

- Iteration i requires i comparisons.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) \sim N^2 / 2$ compares.



Average case. [random]

- Iteration i requires $i / 2$ comparisons on average.
- Total = $(0 + 1 + 2 + ... + N\text{-}1) / 2 \sim N^2 / 4$ compares



Sorting Challenge 1

Q. A credit card company sorts 10 million customer account numbers, for use with binary search.

Using insertion sort, what kind of computer is needed?

A. Toaster
B. Cell phone
C. Your laptop
D. Supercomputer
E. Google server farm

Insertion Sort: Lesson

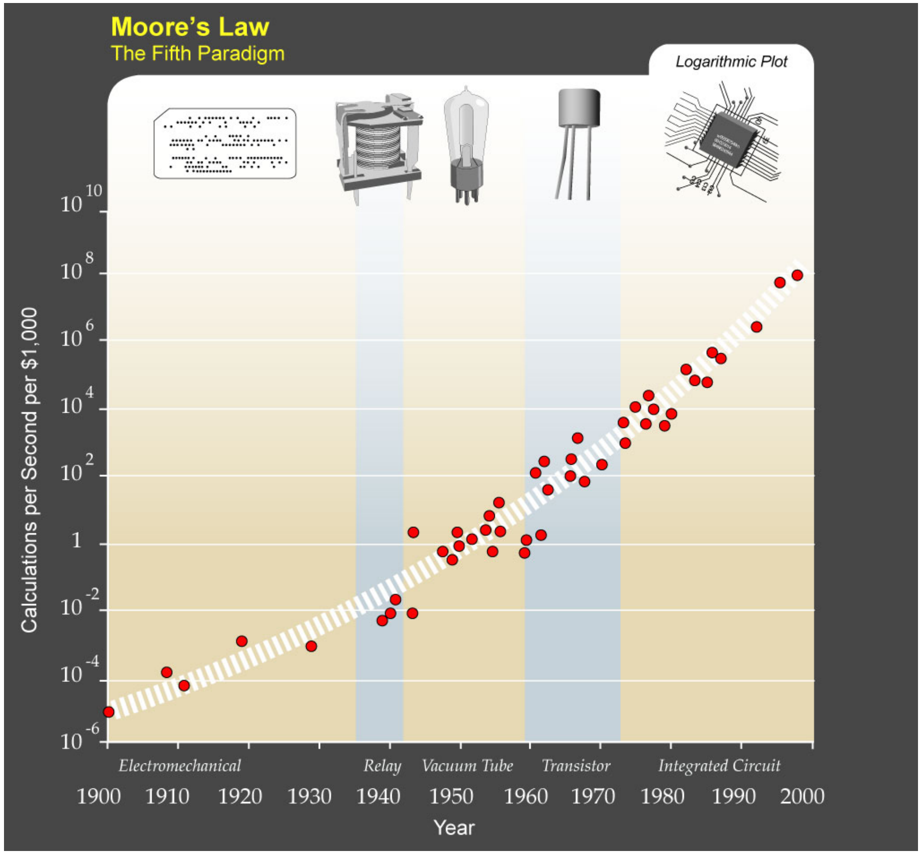
Lesson. Supercomputer can't rescue a bad algorithm.

Computer	Comparisons Per Second	Thousand	Million	Billion
laptop	10^7	instant	1 day	3 centuries
super	10^{12}	instant	1 second	2 weeks

Moore's Law

Moore's law. Transistor density on a chip doubles every 2 years.

Variants. Memory, disk space, bandwidth, computing power per $.



http://en.wikipedia.org/wiki/Moore's_law

Moore's Law and Algorithms

Quadratic algorithms do not scale with technology.

- New computer may be 10x as fast.
- But, has 10x as much memory so problem may be 10x bigger.
- With quadratic algorithm, takes 10x as long!

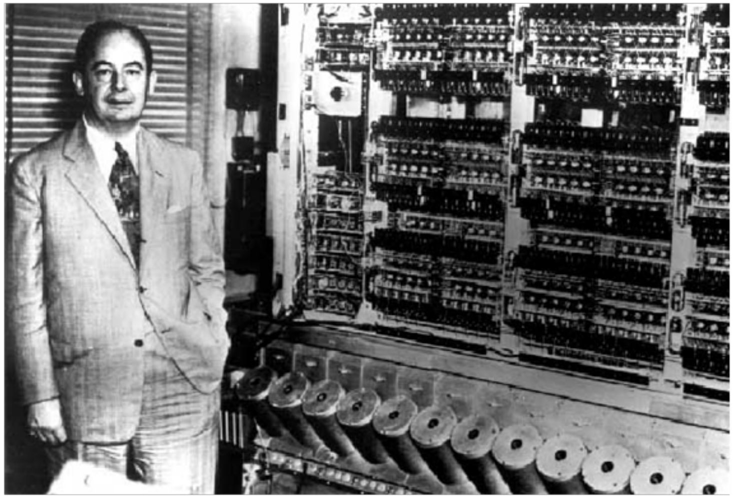
> *"Software inefficiency can always outpace Moore's Law. Moore's Law isn't a match for our bad coding."* – *Jaron Lanier*

Lesson. Need linear (or linearithmic) algorithm to keep pace with Moore's law.

Mergesort

First Draft
of a
Report on the
EDVAC

John von Neumann



Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.

input

`was had him and you his the but`

sort left

`and had him was you his the but`

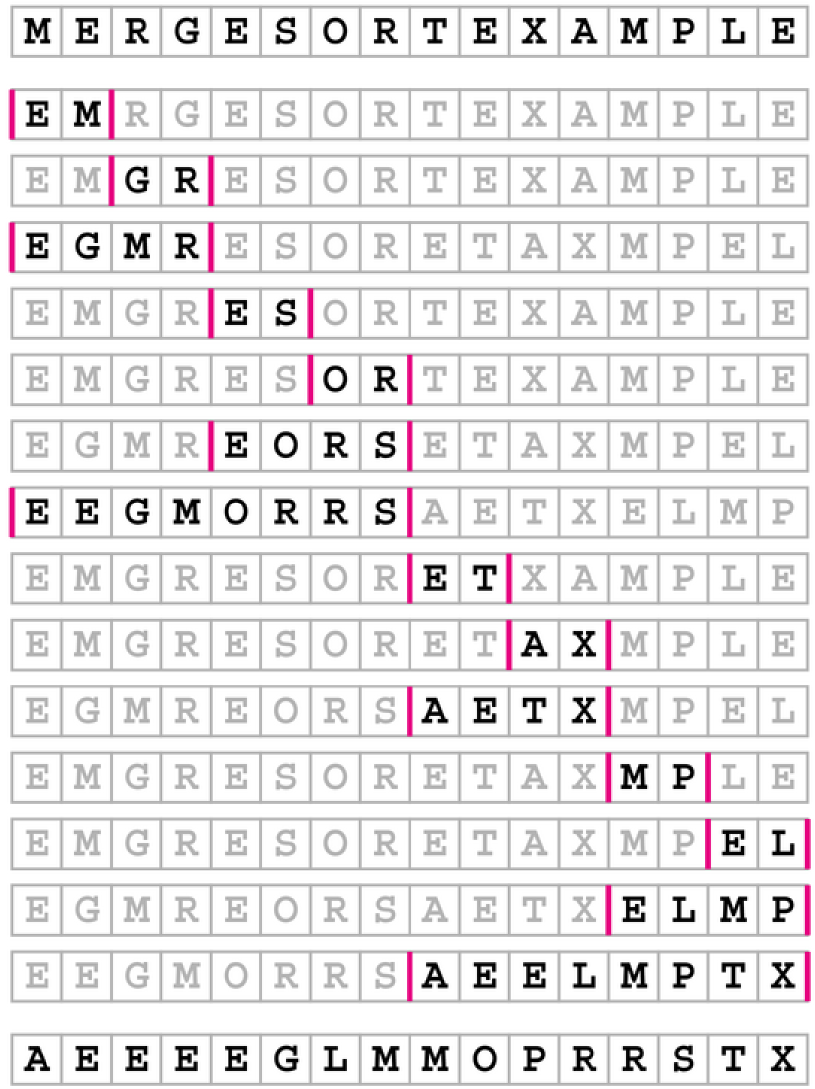
sort right

`and had him was but his the you`

merge

`and but had him his the was you`

Mergesort: Example



Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

i	j	k	aux[k]	a							
				0	1	2	3	4	5	6	7
				and	had	him	was	but	his	the	you
0	4	0	and	and	had	him	was	but	his	the	you
1	4	1	but	and	had	him	was	but	his	the	you
1	5	2	had	and	had	him	was	but	his	the	you
2	5	3	him	and	had	him	was	but	his	the	you
3	5	4	his	and	had	him	was	but	his	the	you
3	6	5	the	and	had	him	was	but	his	the	you
3	6	6	was	and	had	him	was	but	his	the	you
4	7	7	you	and	had	him	was	but	his	the	you

Merging

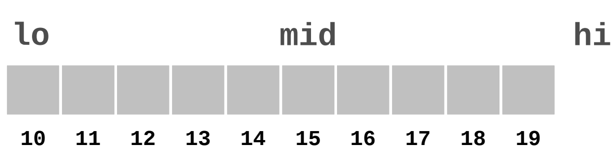
Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently? Use an auxiliary array.

```
String[] aux = new String[N];
// merge into auxiliary array
int i = lo, j = mid;
for (int k = 0; k < N; k++) {
   if      (i == mid) aux[k] = a[j++];
   else if (j == hi)  aux[k] = a[i++];
   else if (a[j].compareTo(a[i]) < 0) aux[k] = a[j++];
   else                               aux[k] = a[i++];
}

// copy back
for (int k = 0; k < N; k++) {
   a[lo + k] = aux[k];
}
```

Mergesort: Java Implementation

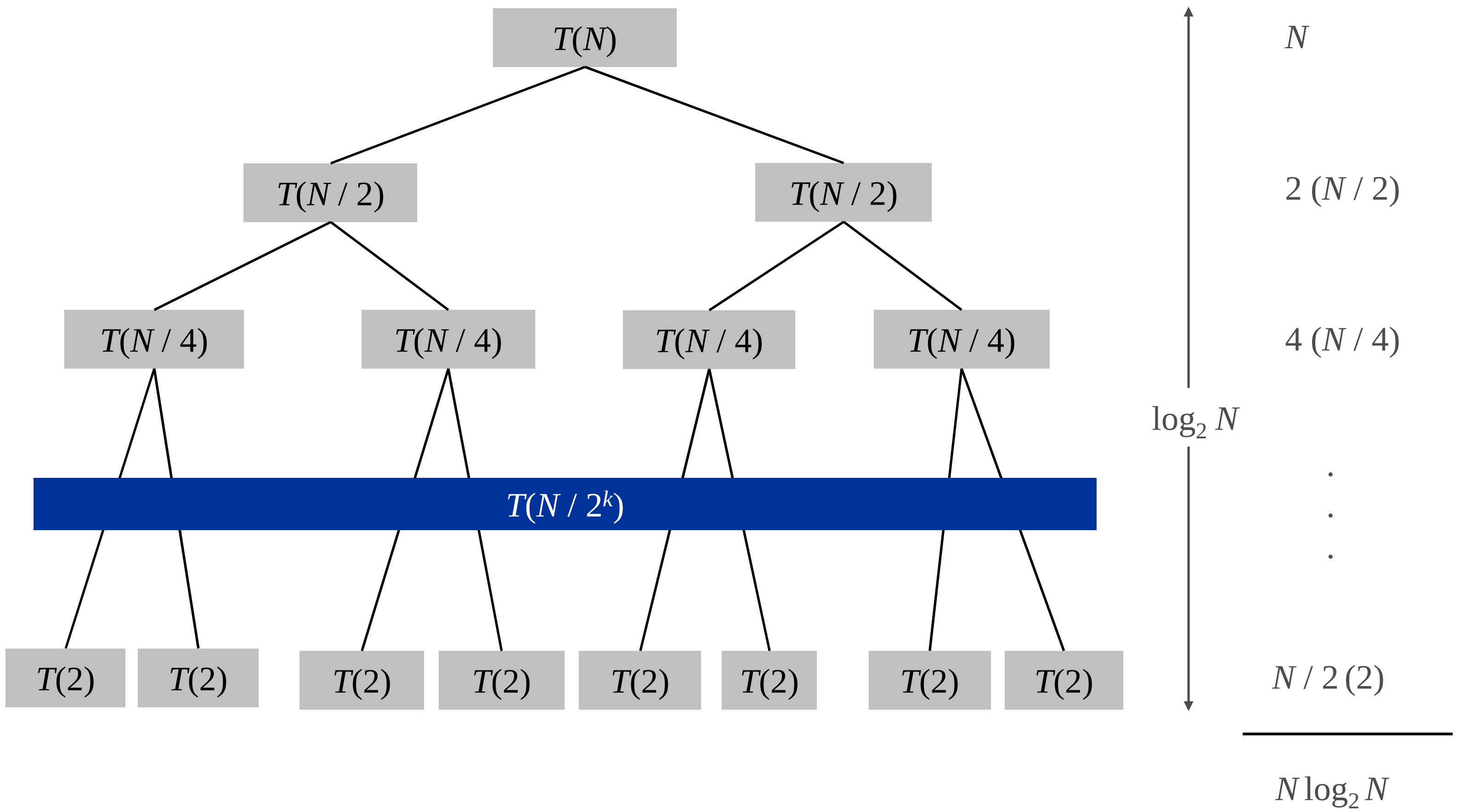
```java
public class Merge {

   public static void sort(String[] a) {
      sort(a, 0, a.length);
   }

   // Sort a[lo, hi).
   public static void sort(String[] a, int lo, int hi) {
      int N = hi - lo;
      if (N <= 1) return;

      // recursively sort left and right halves
      int mid = lo + N/2;
      sort(a, lo, mid);
      sort(a, mid, hi);

      // merge sorted halves (see previous slide)
   }

}
```



Mergesort: Mathematical Analysis

Analysis. To mergesort array of size N, mergesort two subarrays of size $N / 2$, and merge them together using $\leq N$ comparisons.

we assume N is a power of 2



Mergesort: Mathematical Analysis

Mathematical analysis.

analysis	comparisons
worst	$N \log_2 N$
average	$N \log_2 N$
best	$1/2\ N \log_2 N$

Validation. Theory agrees with observations.

N	actual	predicted
10,000	120 thousand	133 thousand
20 million	460 million	485 million
50 million	1,216 million	1,279 million

Sorting Challenge 2

Q. A credit card company sorts 10 million customer account numbers, for use with binary search.

Using mergesort, what kind of computer is needed?

A. Toaster
B. Cell phone
C. Your laptop
D. Supercomputer
E. Google server farm

Sorting Challenge 3

Q. What's the fastest way to sort 1 million 32-bit integers?

Mergesort: Lesson

Lesson. Great algorithms can be more powerful than supercomputers.

Computer	Compares Per Second	Insertion	Mergesort
laptop	10^7	3 centuries	3 hours
super	10^{12}	2 weeks	instant

N = 1 billion

Longest Repeated Substring

Redundancy Detector

Longest repeated substring. Given a string, find the longest substring that appears at least twice.

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c

Brute force.

- Try all indices i and j for start of possible match.
- Compute longest common prefix for each pair (quadratic+).

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c
	i →								j →					

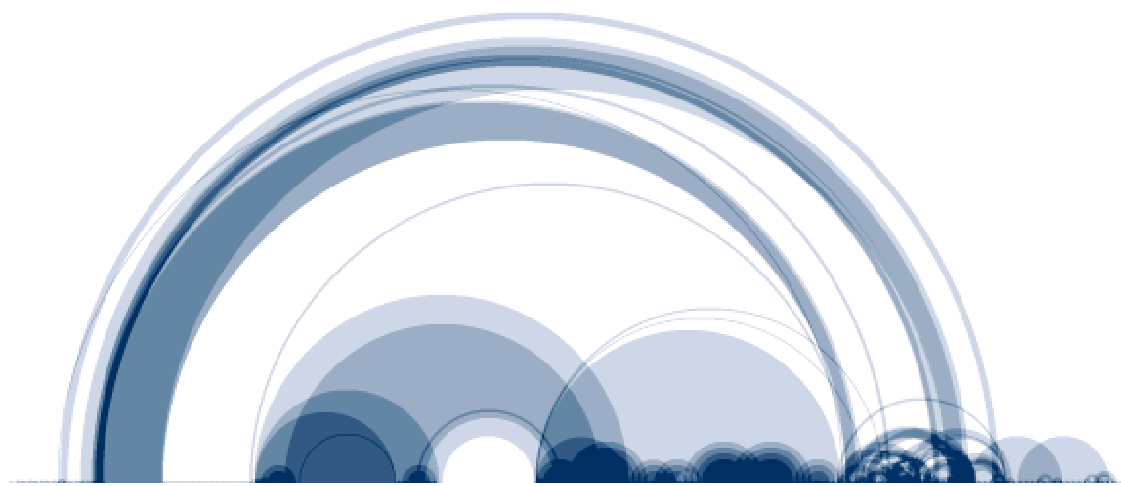
Applications. Bioinformatics, data compression, …

LRS Application: The Shape of a Song

Music is characterized by its repetitive structure.

Mary Had a Little Lamb

Like a Prayer



http://www.bewitched.com

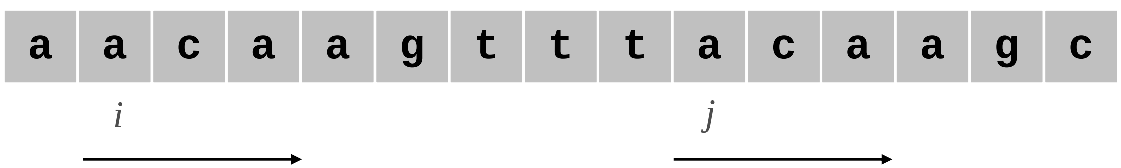
Longest Repeated Substring: Brute-Force Solution

Longest repeated substring. Given a string, find the longest substring that appears at least twice.

a	a	c	a	a	g	t	t	t	a	c	a	a	g	c

Brute force.

- Try all indices *i* and *j* for start of possible match.
- Compute longest common prefix (LCP) for each pair.



Mathematical analysis.

- All pairs: 0 + 1 + 2 + … + N-1 ~ $N^2/2$ calls on LCP.
- Way too slow for long strings.

Longest Repeated Substring: A Sorting Solution

input string

```
0 1 2 3 4 5 6 7 8 91011121314
a a c a a g t t t a c a a g c
```

form suffixes

suffixes

```
 0  a a c a a g t t t a c a a g c
 1  a c a a g t t t a c a a g c
 2  c a a g t t t a c a a g c
 3  a a g t t t a c a a g c
 4  a g t t t a c a a g c
 5  g t t t a c a a g c
 6  t t t a c a a g c
 7  t t a c a a g c
 8  t a c a a g c
 9  a c a a g c
10  c a a g c
11  a a g c
12  a g c
13  g c
14  c
```

sort suffixes to bring repeated substrings together

sorted suffixes

```
 0  a a c a a g t t t a c a a g c
11  a a g c
 3  a a g t t t a c a a g c
 9  a c a a g c
 1  a c a a g t t t a c a a g c
12  a g c
 4  a g t t t a c a a g c
14  c
10  c a a g c
 2  c a a g t t t a c a a g c
13  g c
 5  g t t t a c a a g c
 8  t a c a a g c
 7  t t a c a a g c
 6  t t t a c a a g c
```

compute longest prefix between adjacent suffixes

longest repeated substring

```
  1               9
a a c a a g t t t a c a a g c
```

Longest Repeated Substring: Java Implementation

Suffix sorting implementation.

```
int N = s.length();
String[] suffixes = new String[N];
for (int i = 0; i < N; i++)
   suffixes[i] = s.substring(i, N);
Arrays.sort(suffixes);
```

Longest common prefix. `lcp(s, t)`

- Longest string that is a prefix of both s and t.
- Ex: `lcp("acaagtttac", "acaagc") = "acaag"`.
  Easy to implement (you could write this one).

Longest repeated substring. Search only adjacent suffixes.

```
String lrs = "";
for (int i = 0; i < N-1; i++) {
   String x = lcp(suffixes[i], suffixes[i+1]);
   if (x.length() > lrs.length()) lrs = x;
}
```

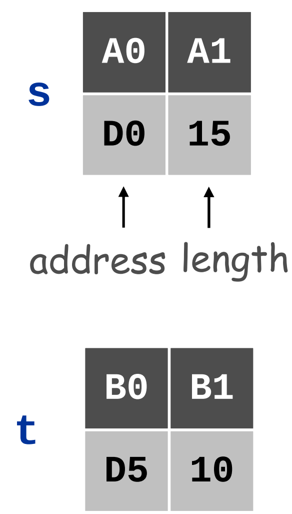
OOP Context for Strings

String representation.

- A `String` is an address and a length.
- Characters can be shared among strings.
- `substring()` computes address and length. ← does not copy chars

D0	D1	D2	D3	D4	D5	D6	D7	D8	D9	DA	DB	DC	DD	DE
a	a	c	a	a	g	t	t	t	a	c	a	a	g	c

```
s = "aacaagtttacaagc";

t = s.substring(5, 15);
```



Consequences.

- `substring()` is constant-time operation (instead of linear).
- Creating suffixes takes linear space (instead of quadratic).
- Running time of LRS is dominated by the string sort.

Sorting Challenge 4

Q. Four researchers A, B, C, and D are looking for long repeated sequences in a genome with over 1 billion characters.

Which one is more likely to find a cure for cancer?

A. has a grad student to do it.
B. uses brute force (check all pairs) solution.
C. uses sorting solution with insertion sort.
D. uses sorting solution with mergesort.

Longest Repeated Substring: Empirical Analysis

Input File	Characters	Brute	Suffix Sort	Length
LRS.java	2,162	0.6 sec	0.14 sec	73
Amendments	18,369	37 sec	0.25 sec	216
Aesop's Fables	191,945	3958 sec	1.0 sec	58
Moby Dick	1.2 million	43 hours †	7.6 sec	79
Bible	4.0 million	20 days †	34 sec	11
Chromosome 11	7.1 million	2 months †	61 sec	12,567
Pi	10 million	4 months †	84 sec	14

† estimated

Lesson. Sorting to the rescue; enables new research.

Summary

Binary search. Efficient algorithm to search a sorted array.

Mergesort. Efficient algorithm to sort an array.

Applications. Many many applications are enabled by fast sorting and searching.

Extra Slides

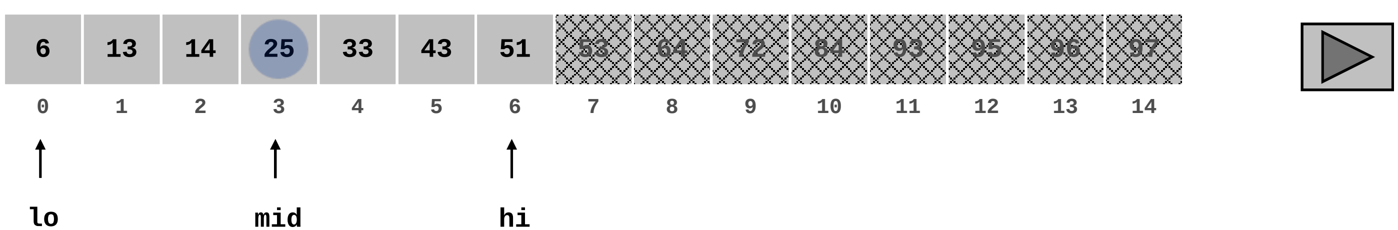
Searching a Sorted Array

Searching a sorted array. Given a sorted array, determine the index associated with a given key.

Ex. Dictionary, phone book, book index, credit card numbers, …

Binary search.

- Examine the middle key.
- If it matches, return its index.
- Otherwise, search either the left or right half.



Binary Search: Nonrecursive Implementation

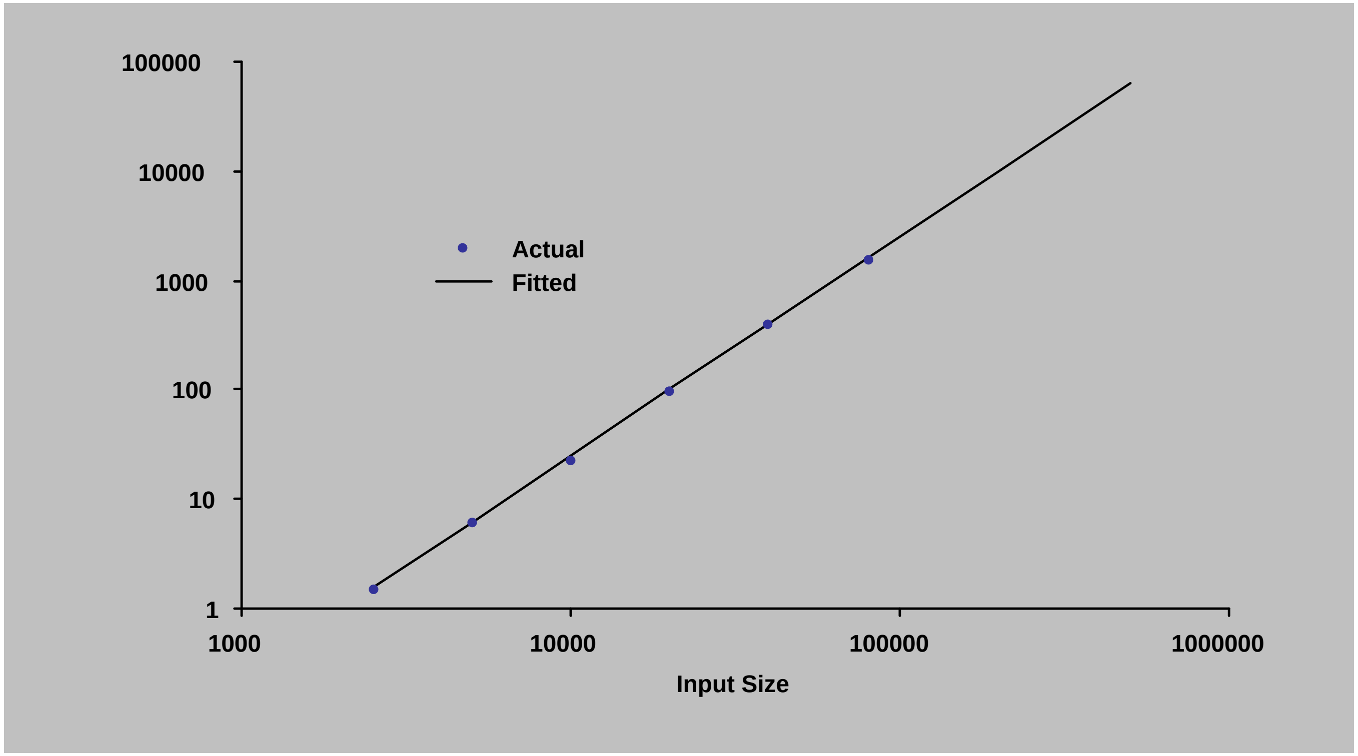
Invariant. *Algorithm maintains* `a[lo]` ≤ `key` ≤ `a[hi]`.

```
public static int search(String[] a, String key) {
   int lo = 0;
   int hi = N-1;
   while (lo <= hi) {
      int mid = lo + (hi - lo) / 2;
      int cmp = key.compareTo(a[mid]);
      if      (cmp < 0) hi = mid - 1;
      else if (cmp > 0) lo = mid + 1;
      else return mid;
   }
   return -1;
}
```

Java library implementation: `Arrays.binarySearch()`

Insertion Sort: Empirical Analysis

Data analysis. Plot # comparisons vs. input size on log-log scale.



slope

Hypothesis. # comparisons grows quadratically with input size ~ $N^2 / 4$.

Insertion Sort: Observation

Observe and tabulate running time for various values of N.

- Data source: N random numbers between 0 and 1.
- Machine: Apple G5 1.8GHz with 1.5GB memory running OS X.
- Timing: Skagen wristwatch.

N	Comparisons	Time
5,000	6.2 million	0.13 seconds
10,000	25 million	0.43 seconds
20,000	99 million	1.5 seconds
40,000	400 million	5.6 seconds
80,000	1600 million	23 seconds

Insertion Sort: Prediction and Verification

Experimental hypothesis. # comparisons ~ $N^2/4$.

Prediction. 400 million comparisons for N = 40,000.

Observations.

N	Comparisons	Time
40,000	401.3 million	5.595 sec
40,000	399.7 million	5.573 sec
40,000	401.6 million	5.648 sec
40,000	400.0 million	5.632 sec

Agrees.

Prediction. 10 billion comparisons for N = 200,000.

Observation.

N	Comparisons	Time
200,000	9.997 billion	145 seconds

Agrees.

Insertion Sort: Mathematical Analysis

Mathematical analysis.

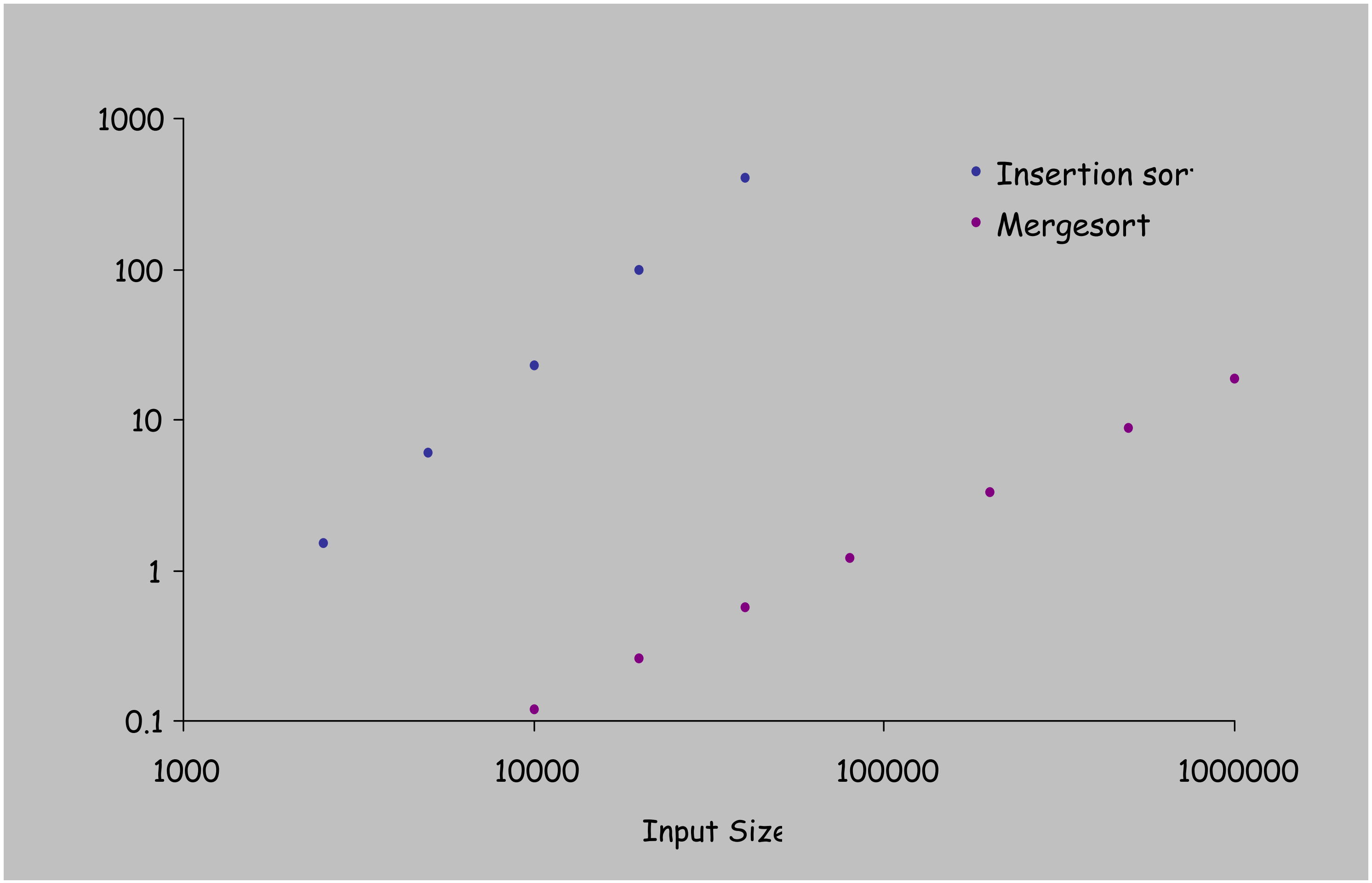
Analysis	Comparisons	Stddev
Worst	$N^2 / 2$	-
Average	$N^2 / 4$	$1/6\ N^{3/2}$
Best	N	-

Validation. Theory agrees with observations.

N	Actual	Predicted
40,000	401.3 million	400 million
200,000	9.9997 billion	10.000 billion

Mergesort: Preliminary Hypothesis

Experimental hypothesis. Number of comparisons ~ 20N.



Mergesort: Prediction and Verification

Experimental hypothesis. Number of comparisons ~ 20N.

Prediction. 80 million comparisons for N = 4 million.

Observations.

N	Comparisons	Time
4 million	82.7 million	3.13 sec
4 million	82.7 million	3.25 sec
4 million	82.7 million	3.22 sec

Agrees.

Prediction. 400 million comparisons for N = 20 million.

Observations.

N	Comparisons	Time
20 million	460 million	17.5 sec
50 million	1216 million	45.9 sec

Not quite.