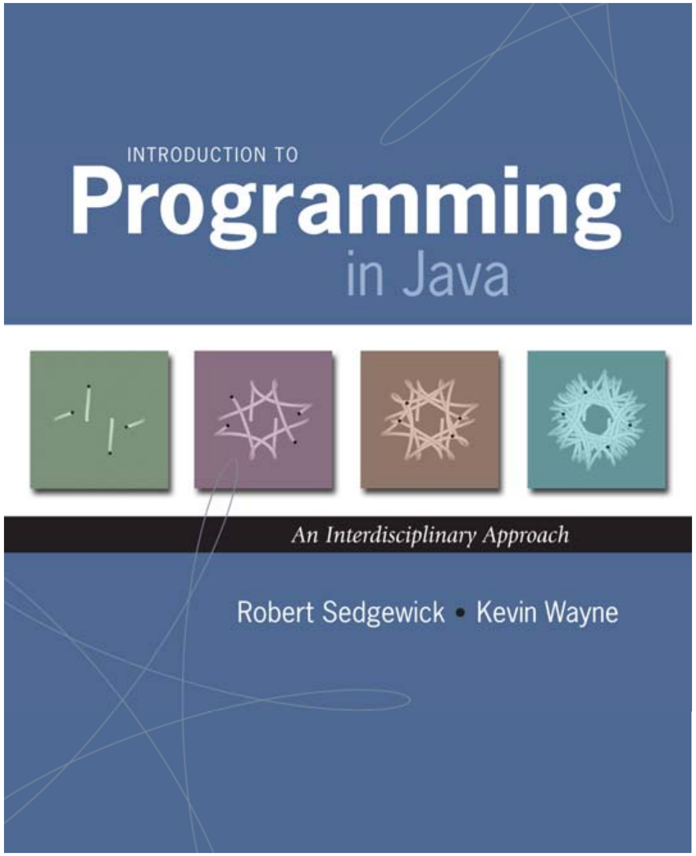
4.1 Performance

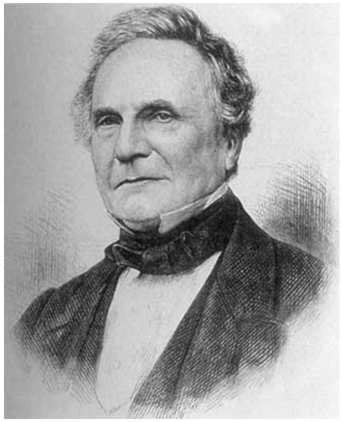

INTRODUCTION TO Programming in Java

An Interdisciplinary Approach

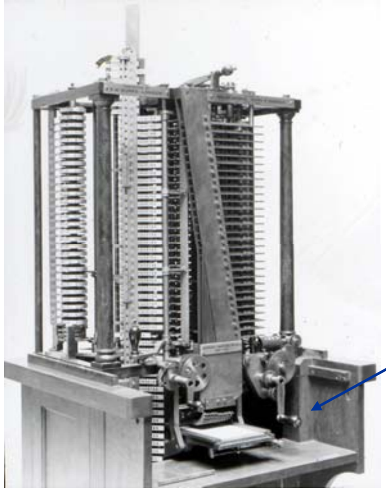
Robert Sedgewick • Kevin Wayne

Running Time

> *"As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time?"* – Charles Babbage



Charles Babbage (1864)



how many times do you have to turn the crank?

Analytic Engine

The Challenge

Q. Will my program be able to solve a large practical problem?

bob pomfret 2005

compile

debug on
test case

solve problems
in practice

Key insight. [Knuth 1970s]
Use the scientific method to understand performance.

Scientific Method

Scientific method.

- Observe some feature of the natural world.
- Hypothesize a model that is consistent with the observations.
- Predict events using the hypothesis.
- Verify the predictions by making further observations.
- Validate by repeating until the hypothesis and observations agree.

Principles.

- Experiments we design must be reproducible.
- Hypothesis must be falsifiable.

Reasons to Analyze Algorithms

Predict performance.

- Will my program finish?
- When will my program finish?

Compare algorithms.

- Will this change make my program faster?
- How can I make my program faster?

Basis for inventing new ways to solve problems.

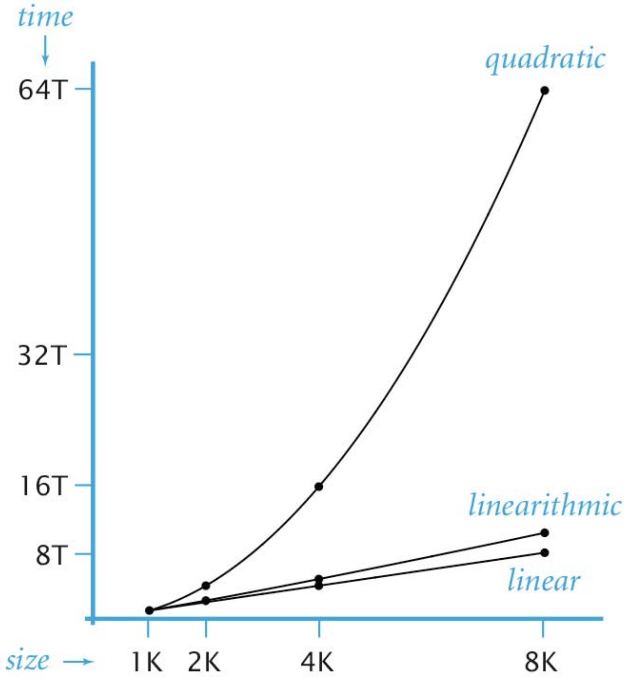
- Enables new technology.
- Enables new research.

Algorithmic Successes

Freidrich Gauss
1805

Discrete Fourier transform.

- Break down waveform of N samples into periodic components.
- Applications: DVD, JPEG, MRI, astrophysics, ….
- Brute force: N^2 steps.
- FFT algorithm: N log N steps, enables new technology.

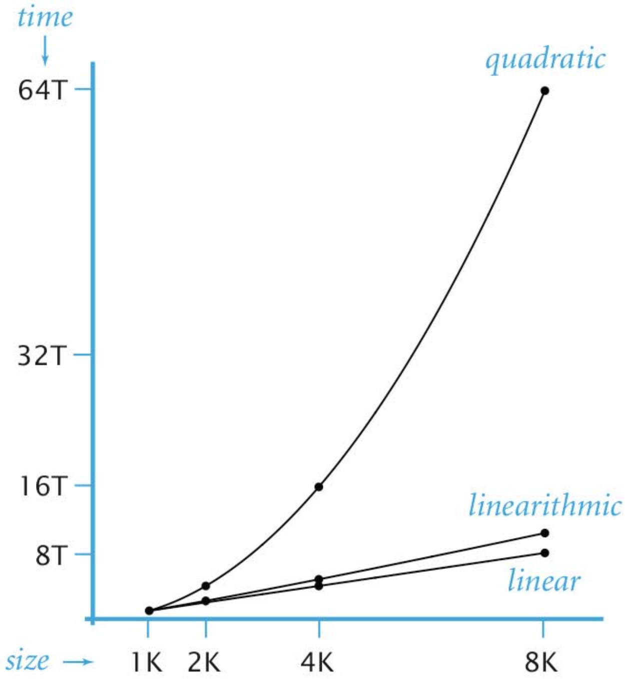


Algorithmic Successes

Andrew Appel
PU '81

N-body Simulation.

- Simulate gravitational interactions among N bodies.
- Brute force: N^2 steps.
- Barnes-Hut: N log N steps, enables new research.



Three-Sum Problem

Three-sum problem. Given N integers, find triples that sum to 0.

Context. Deeply related to problems in computational geometry.

```
% more 8ints.txt
30 -30 -20 -10 40 0 10 5

% java ThreeSum < 8ints.txt
 4
 30 -30   0
 30 -20 -10
-30 -10  40
-10   0  10
```

Q. How would you write a program to solve the problem?

Three-Sum: Brute-Force Solution

```
public class ThreeSum {

   // return number of distinct triples (i, j, k)
   // such that (a[i] + a[j] + a[k] == 0)
   public static int count(int[] a) {
      int N = a.length;
      int cnt = 0;
      for (int i = 0; i < N; i++)
         for (int j = i+1; j < N; j++)
            for (int k = j+1; k < N; k++)
               if (a[i] + a[j] + a[k] == 0) cnt++;
      return cnt;
   }

   public static void main(String[] args) {
      int[] a = StdArrayIO.readInt1D();
      StdOut.println(count(a));
   }
}
```

Empirical Analysis

Empirical Analysis

Empirical analysis. Run the program for various input sizes.

N	*time* †
512	0.03
1024	0.26
2048	2.16
4096	17.18
8192	136.76

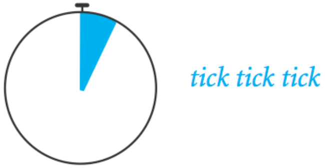
† Running Linux on Sun-Fire-X4100 with 16GB RAM

Stopwatch

Q. How to time a program?

A. A stopwatch.

```
% java ThreeSum < 1Kints.txt
```



```
0
% java ThreeSum < 2Kints.txt
```

```
2
391930676 -763182495 371251819
-326747290 802431422 -475684132
```

Stopwatch

Q. How to time a program?

A. A `Stopwatch` object.

`public class Stopwatch`

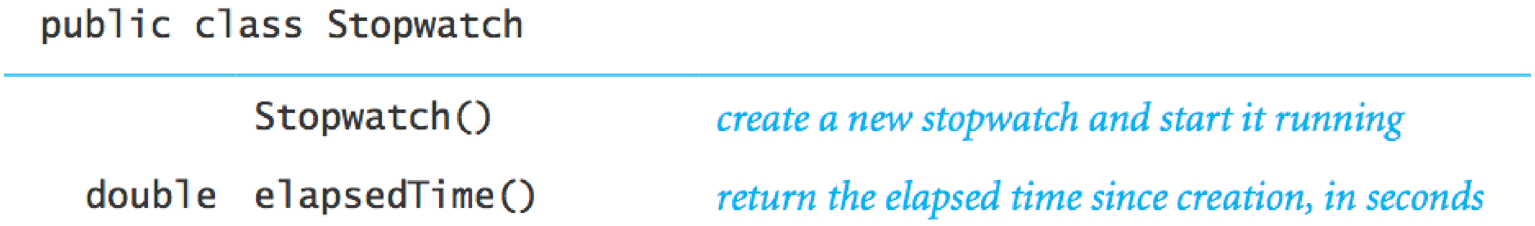
	`Stopwatch()`	*create a new stopwatch and start it running*
`double`	`elapsedTime()`	*return the elapsed time since creation, in seconds*

```java
public class Stopwatch {
   private final long start;

   public Stopwatch() {
      start = System.currentTimeMillis();
   }

   public double elapsedTime() {
      return (System.currentTimeMillis() - start) / 1000.0;
   }
}
```

Stopwatch

Q. How to time a program?

A. A **Stopwatch** object.



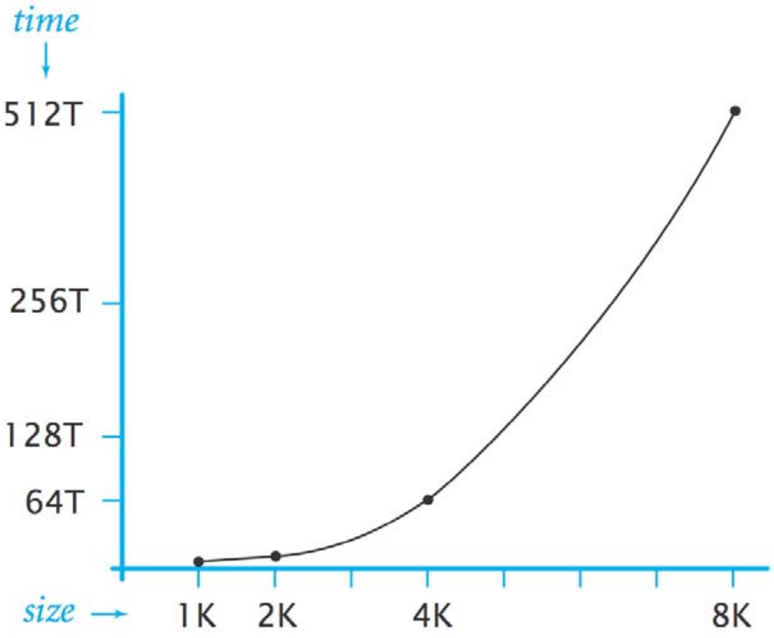
`public class Stopwatch`

	`Stopwatch()`	*create a new stopwatch and start it running*
`double`	`elapsedTime()`	*return the elapsed time since creation, in seconds*

```
public static void main(String[] args) {
   int[] a = StdArrayIO.readInt1D();
   Stopwatch timer = new Stopwatch();
   StdOut.println(count(a));
   StdOut.println(timer.elapsedTime());
}
```

Empirical Analysis

Data analysis. Plot running time vs. input size N.



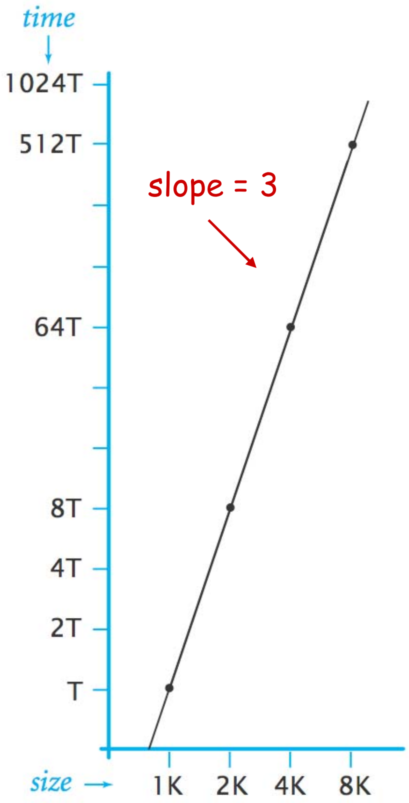
Q. How fast does running time grow as a function of input size N?

Empirical Analysis

Initial hypothesis. Running time obeys power law $f(N) = a\,N^{b}$.

Data analysis. Plot running time vs. input size N on a log-log scale.

Consequence. Power law yields straight line (slope = b).



slope = 3

Refined hypothesis. Running time grows as cube of input size: $a\,N^{3}$.

slope

Doubling Hypothesis

Doubling hypothesis. Quick way to estimate b in a power law hypothesis.

Run program, doubling the size of the input?

N	$time^{\dagger}$	$ratio$
512	0.033	-
1024	0.26	7.88
2048	2.16	8.43
4096	17.18	7.96
8192	136.76	7.96

↑ seems to converge to a constant $c = 8$

Hypothesis. Running time is about $a N^b$ with $b = lg\ c$.

Doubling Challenge 1

Let F(N) be running time of `Mystery` as a function of input N.

```
public static void main(String[] args) {
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

Scenario 1. F(2N) / F(N) converges to about 4.

Q. What is order of growth of the running time?

Doubling Challenge 2

Let F(N) be running time of `Mystery` as a function of input N.

```
public static void main(String[] args) {
   ...
   int N = Integer.parseInt(args[0]);
   ...
}
```

Scenario 2. F(2N) / F(N) converges to about 2.

Q. What is order of growth of the running time?

Prediction and Validation

Hypothesis. Running time is about $a N^3$ for input of size N.

Q. How to estimate a?

A. Run the program!

N	*time* †
4096	17.18
4096	17.15
4096	17.17

$17.17 = a\ 4096^3$

$\Rightarrow a = 2.5 \times 10^{-10}$

Refined hypothesis. Running time is about $2.5 \times 10^{-10} \times N^3$ seconds.

Prediction. 1,100 seconds for $N = 16{,}384$.

Observation.

N	*time* †
16384	1118.86

← *validates hypothesis!*

Mathematical Analysis

Donald Knuth
Turing award '74

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
for (int i = 0; i < N; i++)
   if (a[i] == 0) count++;
```

operation	frequency
variable declaration	2
variable assignment	2
less than comparison	$N + 1$
equal to comparison	N
array access	N
increment	$\le 2N$

between N (no zeros)
and 2N (all zeros)

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.

```
int count = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         if (a[i] + a[j] == 0) count++;
```

operation	frequency
variable declaration	$N + 2$
variable assignment	$N + 2$
less than comparison	$1/2\ (N + 1)\ (N + 2)$
equal to comparison	$1/2\ N\ (N - 1)$
array access	$N\ (N - 1)$
increment	$\leq\ N^2$

$$0 + 1 + 2 + \ldots + (N - 1) = 1/2\ N(N - 1)$$

becoming very tedious to count

Tilde Notation

Tilde notation.

- Estimate running time as a function of input size N.
- Ignore lower order terms.
  - when N is large, terms are negligible
  - when N is small, we don't care

Ex 1. $6N^3 + 17N^2 + 56 \sim 6N^3$

Ex 2. $6N^3 + 100N^{4/3} + 56 \sim 6N^3$

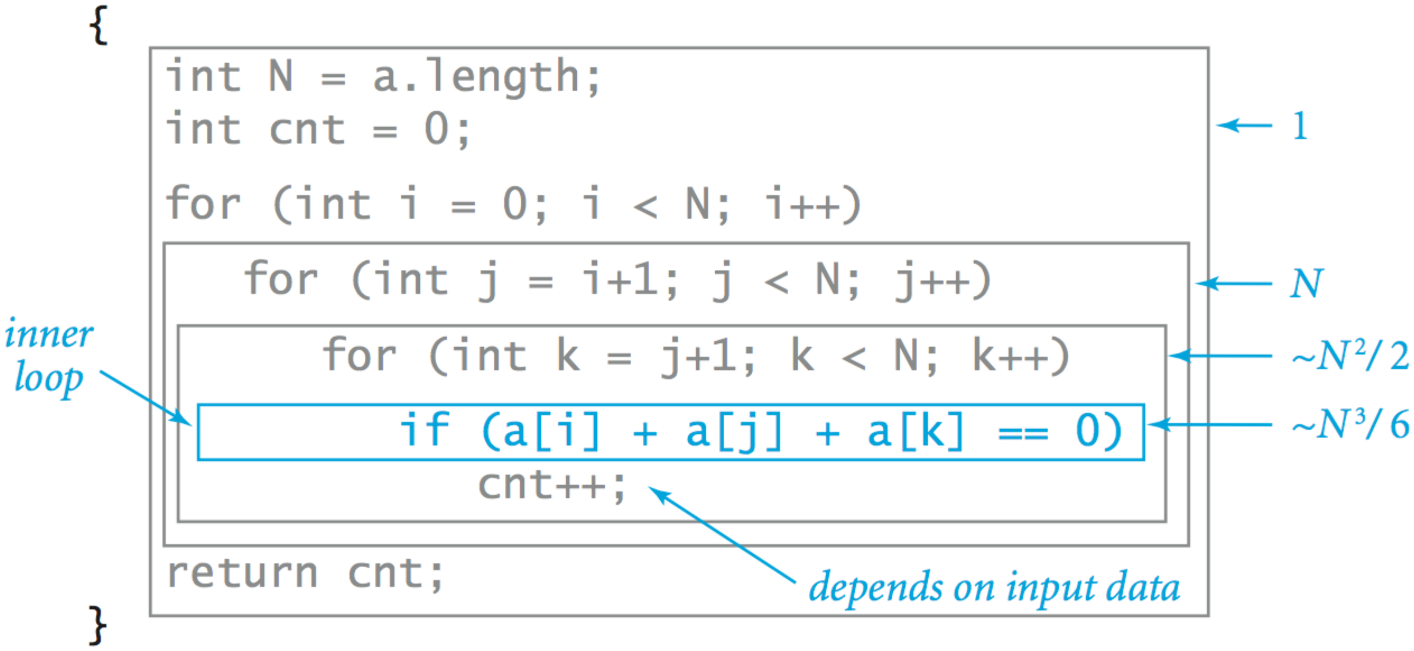
Ex 3. $6N^3 + \underbrace{17N^2 \log N}_{\text{discard lower-order terms}} \sim 6N^3$

discard lower-order terms
(e.g., N = 1000: 6 trillion vs. 169 million)

Technical definition. $f(N) \sim g(N)$ means $\lim_{N \to \infty} \frac{f(N)}{g(N)} = 1$

Mathematical Analysis

Running time. Count up frequency of execution of each instruction and weight by its execution time.



```
{
   int N = a.length;
   int cnt = 0;
   for (int i = 0; i < N; i++)
      for (int j = i+1; j < N; j++)
         for (int k = j+1; k < N; k++)
            if (a[i] + a[j] + a[k] == 0)
               cnt++;
   return cnt;
}
```

Inner loop. Focus on instructions in "inner loop."

Constants in Power Law

Power law. Running time of a typical program is $\sim a\,N^b$.

Exponent a depends on: algorithm.

Constant c depends on:

- algorithm
- input data

system independent effects

- caching
- machine
- compiler
- garbage collection
- just-in-time compilation
- CPU use by other applications

system dependent effects

Our approach. Use doubling hypothesis (or mathematical analysis) to estimate exponent b, run experiments to estimate a.

Analysis: Empirical vs. Mathematical

Empirical analysis.

- Measure running times, plot, and fit curve.
- Easy to perform experiments.
- Model useful for predicting, but not for explaining.

Mathematical analysis.

- Analyze algorithm to estimate # ops as a function of input size.
- May require advanced mathematics.
- Model useful for predicting and explaining.

Critical difference. Mathematical analysis is independent of a particular machine or compiler; applies to machines not yet built.

Order of Growth Classifications

Observation. A small subset of mathematical functions suffice to describe running time of many fundamental algorithms.

```
while (N > 1) {
   N = N / 2;
   ...
}
```

$\lg N$

$\lg N = \log_2 N$

```
for (int i = 0; i < N; i++)
   ...
```

N

```
for (int i = 0; i < N; i++)
   for (int j = 0; j < N; j++)
      ...
```

N^2

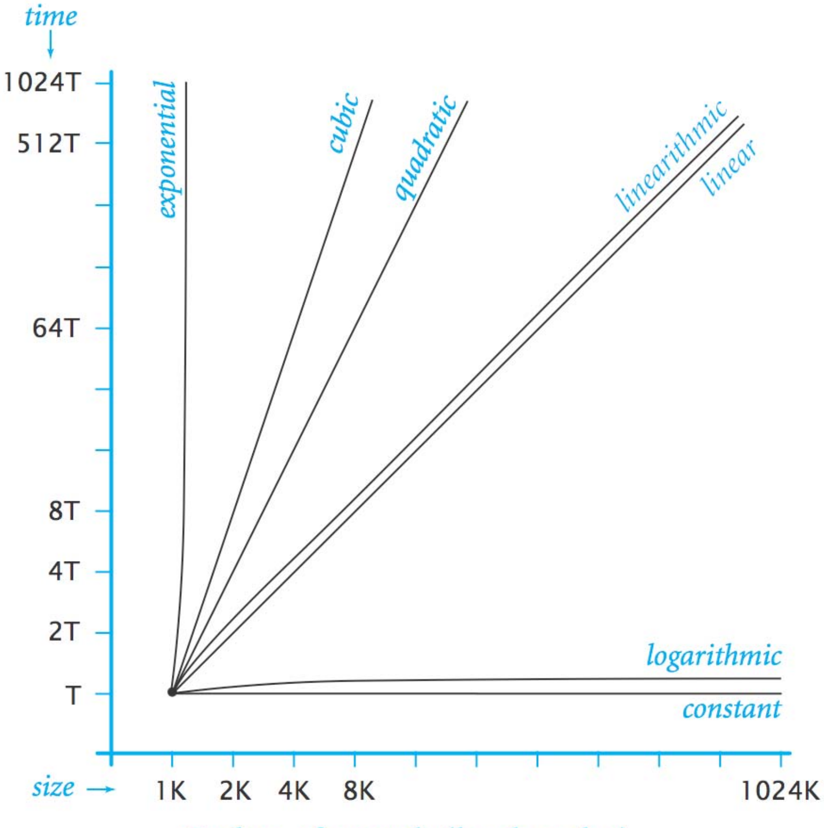
```
public static void g(int N) {
   if (N == 0) return;
   g(N/2);
   g(N/2);
   for (int i = 0; i < N; i++)
      ...
}
```

$N \lg N$

```
public static void f(int N) {
   if (N == 0) return;
   f(N-1);
   f(N-1);
   ...
}
```

2^N

Order of Growth Classifications



Orders of growth (log-log plot)

order of growth		factor for doubling hypothesis
description	function	
constant	1	1
logarithmic	$\log N$	1
linear	N	2
linearithmic	$N \log N$	2
quadratic	N^2	4
cubic	N^3	8
exponential	2^N	2^N

Order of Growth: Consequences

order of growth	predicted running time if problem size is increased by a factor of 100
linear	a few minutes
linearithmic	a few minutes
quadratic	several hours
cubic	a few weeks
exponential	forever

Effect of increasing problem size for a program that runs for a few seconds

order of growth	predicted factor of problem size increase if computer speed is increased by a factor of 10
linear	10
linearithmic	10
quadratic	3-4
cubic	2-3
exponential	1

Effect of increasing computer speed on problem size that can be solved in a fixed amount of time

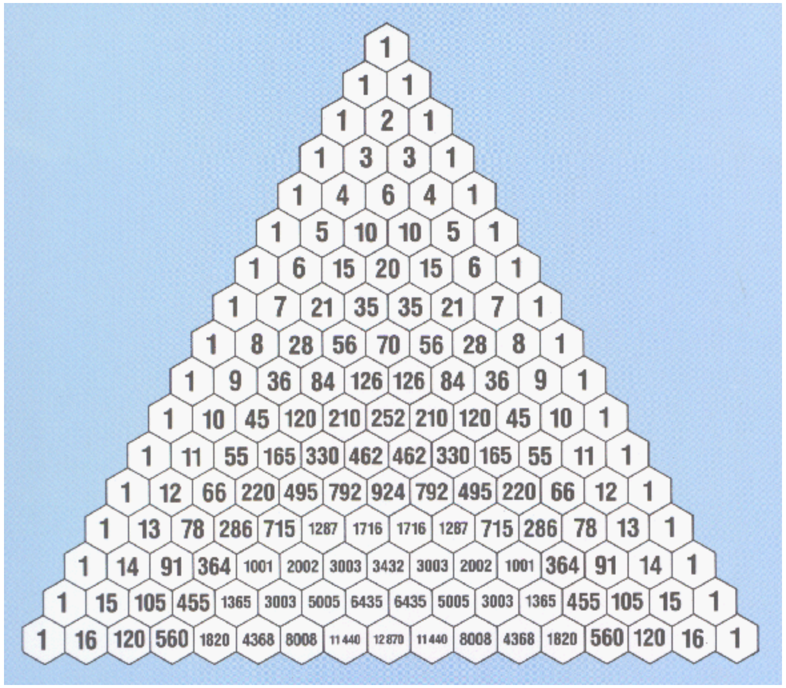
Dynamic Programming

Binomial Coefficients

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Pascal's identity.

$$\binom{n}{k} = \underbrace{\binom{n-1}{k-1}}_{\text{contains first element}} + \underbrace{\binom{n-1}{k}}_{\text{excludes first element}}$$

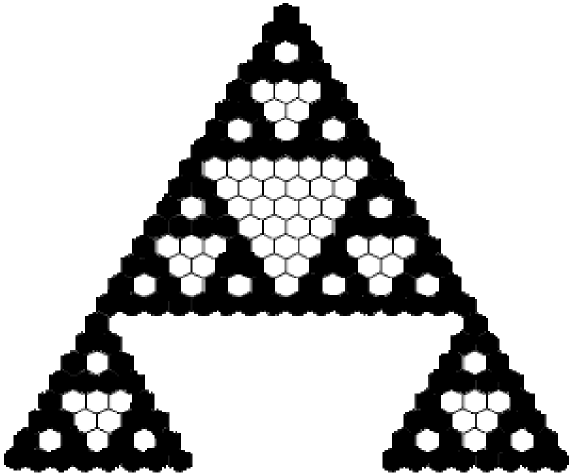


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1
1 16 120 560 1820 4368 8008 11440 12870 11440 8008 4368 1820 560 120 16 1

Binomial Coefficients: Sierpinski Triangle

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Sierpinski triangle. Color black the odd integers in Pascal's triangle.

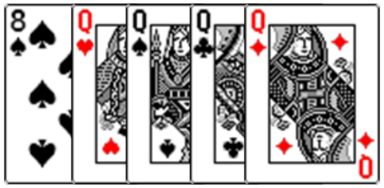


Binomial Coefficients: Poker Odds

Binomial coefficient. $\binom{n}{k}$ = number of ways to choose k of n elements.

Probability of "quads" in Texas hold 'em:

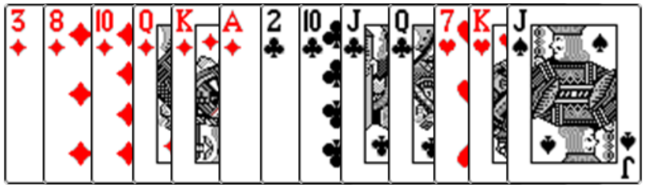
$$\frac{\binom{13}{1} \times \binom{48}{3}}{\binom{52}{7}} = \frac{224{,}848}{133{,}784{,}560} \quad (\textit{about} \ \ 594:1)$$



Probability of 6-4-2-1 split in bridge:

$$\frac{\binom{4}{1} \times \binom{13}{6} \times \binom{3}{1} \times \binom{13}{4} \times \binom{2}{1} \times \binom{13}{2} \times \binom{1}{1} \times \binom{13}{1}}{\binom{52}{13}}$$

$$= \frac{29{,}858{,}811{,}840}{635{,}013{,}559{,}600} \quad (\textit{about} \ \ 21:1)$$



Binomial Coefficients: First Attempt

```
public class SlowBinomial {

   // natural recursive implementation
   public static long binomial(long n, long k) {
      if (k == 0) return 1;
      if (n == 0) return 0;
      return binomial(n-1, k-1) + binomial(n-1, k);
   }

   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      StdOut.println(binomial(N, K));
   }

}
```

Timing Experiments

Timing experiments for binomial coefficients via direct recursive solution.

$(2n, n)$	*time* †
(26, 13)	0.46
(28, 14)	1.27
(30, 15)	4.30
(32, 16)	15.69
(34, 17)	57.40
(36, 18)	230.42

increase n by 1, running time increases by about 4x

Q. Is running time linear, quadratic, cubic, exponential in n?

Doubling Challenge 3

Let F(N) be running time to compute `binomial(2N, N)`.

```
public static long binomial(long n, long k) {
   if (k == 0) return 1;
   if (n == 0) return 0;
   return binomial(n-1, k-1) + binomial(n-1, k);
}
```

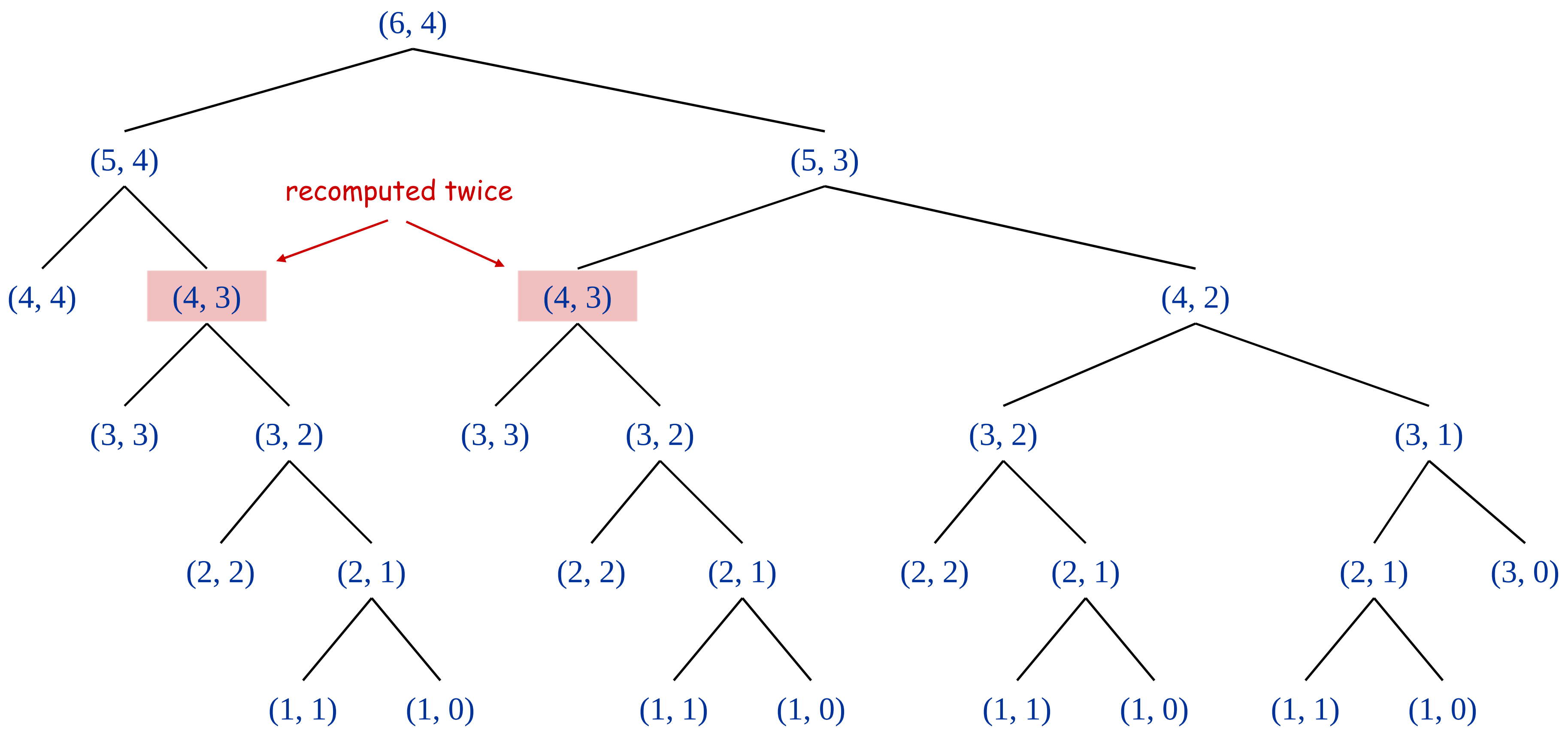
Observation. F(N+1) / F(N) converges to about 4.

Q. What is order of growth of the running time?

A. Exponential: a 4^N. ← will not finish unless N is small

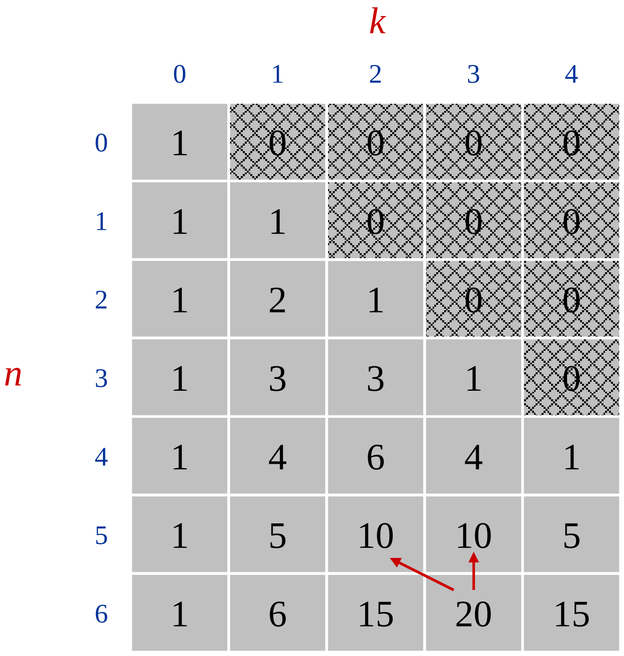
Why So Slow?

Function call tree.



Dynamic Programming

Key idea. Save solutions to subproblems to avoid recomputation.



		k				
		0	1	2	3	4
	0	1	0	0	0	0
	1	1	1	0	0	0
	2	1	2	1	0	0
n	3	1	3	3	1	0
	4	1	4	6	4	1
	5	1	5	10	10	5
	6	1	6	15	20	15

binomial(n, k)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

20 = 10 + 10

Tradeoff. Trade (a little) memory for (a huge amount of) time.

Binomial Coefficients: Dynamic Programming

```
public class Binomial {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int K = Integer.parseInt(args[1]);
      long[][] bin = new long[N+1][K+1];

      // base cases
      for (int k = 1; k <= K; k++) bin[0][K] = 0;
      for (int n = 0; n <= N; n++) bin[N][0] = 1;

      // bottom-up dynamic programming
      for (int n = 1; n <= N; n++)
         for (int k = 1; k <= K; k++)
            bin[n][k] = bin[n-1][k-1] + bin[n-1][k];

      // print results
      StdOut.println(bin[N][K]);
   }
}
```

Timing Experiments

Timing experiments for binomial coefficients via dynamic programming.

(2*n*, *n*)	*time* †
(26, 13)	instant
(28, 14)	instant
(30, 15)	instant
(32, 16)	instant
(34, 17)	instant
(36, 18)	instant

Q. Is running time linear, quadratic, cubic, exponential in n?

Performance Challenge 4

Let F(N) be running time to compute `binomial(2N, N)` using DP.

```
for (int n = 1; n <= N; n++)
   for (int k = 1; k <= K; k++)
      bin[n][k] = bin[n-1][k-1] + bin[n-1][k];
```

Q. What is order of growth of the running time?

A. Quadratic: a N^2. ← effectively instantaneous for small N

Remark. There is a profound difference between 4^N and N^2.

cannot solve
a large problem

can solve
a large problem

Digression: Stirling's Approximation

Alternative:

$$\binom{n}{k} = \frac{n!}{n!\ (n-k)!}$$

Caveat. 52! overflows a long, even though final result doesn't.

Stirling's approximation:

$$\ln n! \approx n \ln n - n + \frac{\ln(2\pi n)}{2} + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}$$

Application. Probability of exact k heads in n flips with a biased coin.

$$\binom{n}{k} p^k \ (1-p)^{n-k}$$

(easy to compute approximate value with Stirling's formula)

Memory

Typical Memory Requirements for Java Data Types

Bit. 0 or 1.

Byte. 8 bits.

Megabyte (MB). 2^{10} bytes ~ 1 million bytes.

Gigabyte (GB). 2^{20} bytes ~ 1 billion bytes.

type	*bytes*
`boolean`	1
`byte`	1
`char`	2
`int`	4
`float`	4
`long`	8
`double`	8

type	*bytes*
`int[]`	$4N + 16$
`double[]`	$8N + 16$
`int[][]`	$4N^2 + 20N + 16$
`double[][]`	$8N^2 + 20N + 16$
`String`	$2N + 40$

typical computer '08 has about 1GB memory

Q. What's the biggest `double` array you can store on your computer?

Performance Challenge 5

Q. How much memory does this program use as a function of `N`?

```
public class RandomWalk {
   public static void main(String[] args) {
      int N = Integer.parseInt(args[0]);
      int[][] count = new int[N][N];
      int x = N/2;
      int y = N/2;

      for (int i = 0; i < N; i++)  {
         // no new variable declared in loop
         ...
         count[x][y]++;
      }
   }
}
```

A.

Summary

Q. How can I evaluate the performance of my program?
A. Computational experiments, mathematical analysis.

Q. What if it's not fast enough? Not enough memory?

- Understand why.
- Buy a faster computer.
- Learn a better algorithm (COS 226, COS 423).
- Discover a new algorithm.

attribute	better machine	better algorithm
cost	$$$ or more	$ or less
applicability	makes "everything" run faster	does not apply to some problems
improvement	quantitative improvements	dramatic qualitative improvements possible