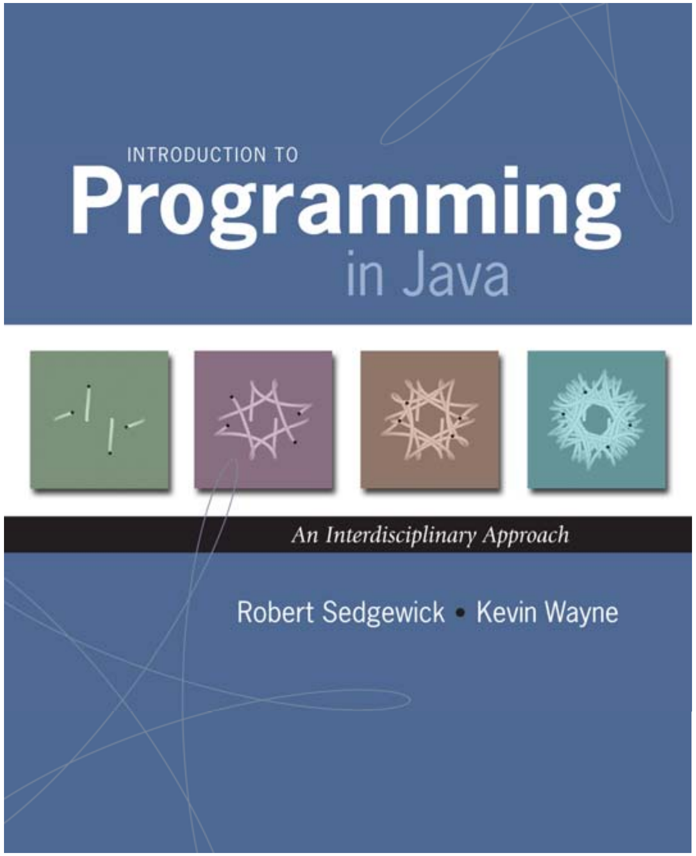
3.4 N-body Simulation



N-Body Problem

Goal. Determine the motion of N particles, moving under their mutual Newtonian gravitational forces.

Ex. Planets orbit the sun.

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

N-Body: Applications

Applications to astrophysics.

- Orbits of solar system bodies.
- Stellar dynamics at the galactic center.
- Stellar dynamics in a globular cluster.
- Stellar dynamics during the collision of two galaxies.
- Formation of structure in the universe.
- Dynamics of galaxies during cluster formation.

N-Body Problem

Goal. Determine the motion of N particles, moving under their mutual Newtonian gravitational forces.

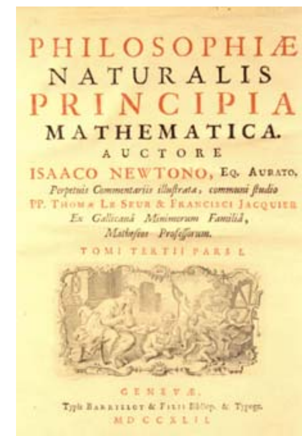
Context. Newton formulated the physical principles in Principia.

$$F = m\,a$$

Newton's second law of motion

$$F = \frac{G\,m_1\,m_2}{r^2}$$

Newton's law of universal gravitation

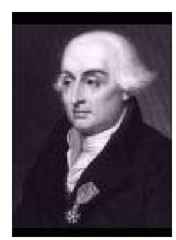

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA.
AUCTORE
ISAACO NEWTONO, Eq. Aurato,
Perpetuis Commentariis illustrata, communi studio
PP. Thomæ Le Seur & Francisci Jacquier
Ex Gallicanâ Minimorum Familiâ,
Matheseos Professorum.
TOMI TERTII PARS I.
GENEVÆ,
MDCCXLII.


Kepler


Bernoulli


Newton


Euler


Lagrange


Delaunay


Poincaré

2-Body Problem

2 body problem.

- Can be solved analytically via Kepler's 3rd law.
- Bodies move around a common barycenter (center-of-mass) with elliptical orbits.

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

3-Body Problem

3-body problem. No solution possible in terms of elementary functions; moreover, orbits may not be stable or periodic!

Consequence. Must resort to computational methods.

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

N-Body Simulation

N-body simulation. The ultimate object-oriented program: simulate the universe.

QuickTime™ and a
H.264 decompressor
are needed to see this picture.

Body Data Type

Body data type. Represent a particle.

public class Body		
	`Body(Vector r, Vector v, double mass)`	
`void`	`move(Vector f, double dt)`	*apply force* `f`*, move body for* `dt` *seconds*
`void`	`draw()`	*draw the ball*
`Vector`	`forceFrom(Body b)`	*force vector between this body and* `b`

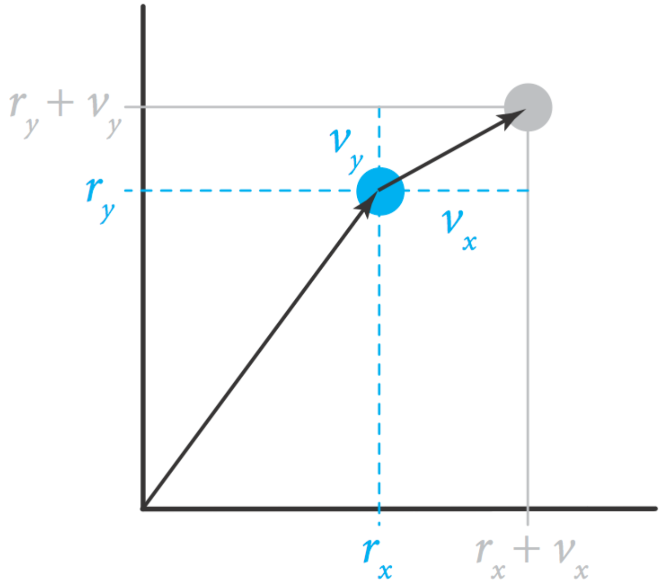
Vector notation. Represent position, velocity, and force using `Vector`.

```
public class Body {
   private Vector r;         // position
   private Vector v;         // velocity
   private double mass;      // mass
```

instance variables

Moving a Body

Moving a body. Assuming no other forces, body moves in straight line.



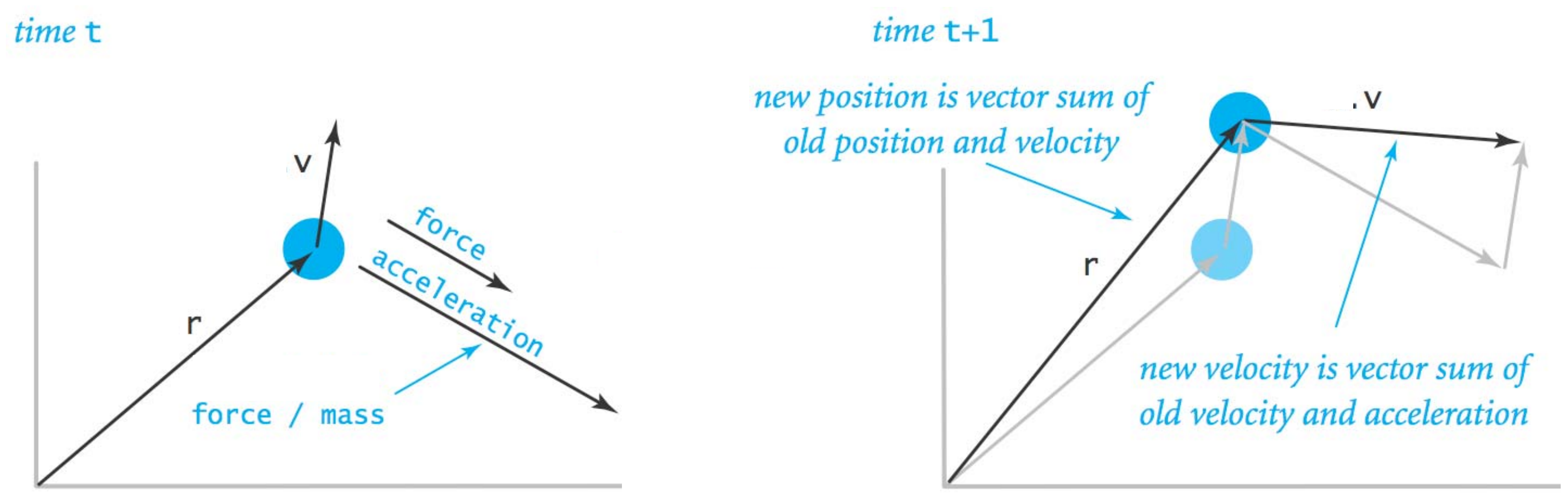

```
r = r.plus(v.times(dt));
```

$$
\begin{aligned}
r_x &= r_x + dt \cdot v_x \\
r_y &= r_y + dt \cdot v_y
\end{aligned}
$$

Moving a Body

Moving a body.

- Given external force F, acceleration a = F/m.
- Use acceleration (assume fixed) to compute change in velocity.
- Use velocity to compute change in position.

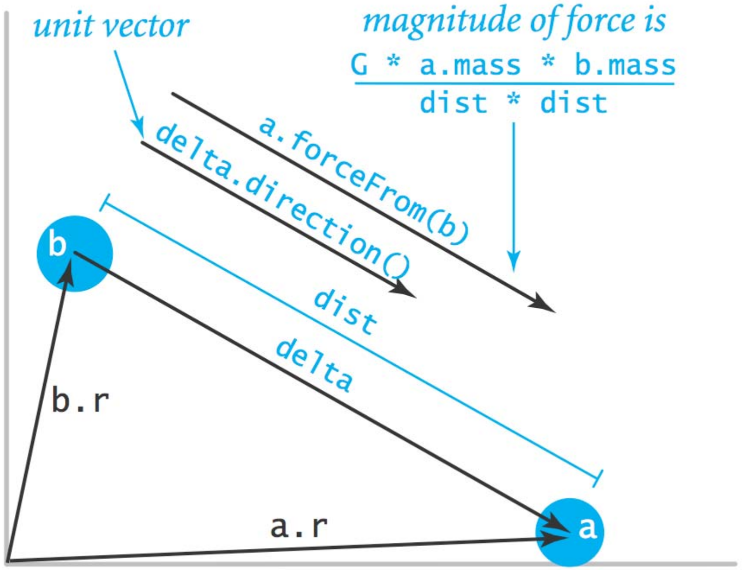



```
Vector a = f.times(1/mass);
v = v.plus(a.times(dt));
r = r.plus(v.times(dt));
```

Force Between Two Bodies

Newton's law of universal gravitation.

- $F = G\, m_1\, m_2 / r^2$.
- Direction of force is line between two particles.




```
double G = 6.67e-11;
Vector delta = a.r.minus(b.r);
double dist = delta.magnitude();
double F = (G * a.mass * b.mass) / (dist * dist);
Vector force = delta.direction().times(F);
```

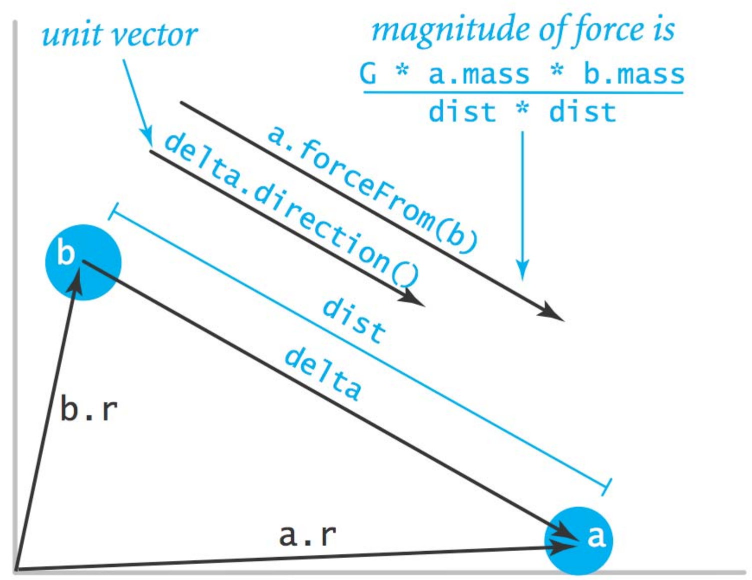
Body Data Type: Java Implementation

```
public class Body {
   private Vector r;      // position
   private Vector v;      // velocity
   private double mass;   // mass

   public Body(Vector r, Vector v, double mass) {
      this.r = r;
      this.v = v;
      this.mass = mass;
   }

   public void move(Vector f, double dt) {
      Vector a = f.times(1/mass);
      v = v.plus(a.times(dt));
      r = r.plus(v.times(dt));
   }

   public Vector forceTo(Body that) {
      double G = 6.67e-11;
      Vector delta = this.r.minus(that.r);
      double dist = delta.magnitude();
      double F = (G * this.mass * that.mass) / (dist * dist);
      return delta.direction().times(F);
   }

   public void draw() {
      StdDraw.setPenRadius(0.025);
      StdDraw.point(r.cartesian(0), r.cartesian(1));
   }
}
```

Force Between Two Bodies

Newton's law of universal gravitation.

- $F = G\, m_1\, m_2 \,/\, r^2$.
- Direction of force is line between two particles.




```
double G = 6.67e-11;
Vector delta = a.r.minus(b.r);
double dist = delta.magnitude();
double F = (G * a.mass * b.mass) / (dist * dist);
Vector force = delta.direction().times(F);
```

Universe Data Type

Universe data type. Represent a universe of N particles.

public class Universe		
	`Universe()`	
`void`	`increaseTime(double dt)`	*simulate the passing of* `dt` *seconds*
`void`	`draw()`	*draw the universe*

```
public static void main(String[] args) {
   Universe newton = new Universe();
   double dt = Double.parseDouble(args[0]);
   while (true) {
      StdDraw.clear();
      newton.increaseTime(dt);
      newton.draw();
      StdDraw.show(10);
   }
}
```

main simulation loop

Universe Data Type

Universe data type. Represent a universe of N particles.

public class Universe

	`Universe()`	
`void`	`increaseTime(double dt)`	*simulate the passing of* `dt` *seconds*
`void`	`draw()`	*draw the universe*

```
public class Universe {
   private double radius; // radius of universe
   private int N          // number of particles
   private Body[] orbs;   // the bodies
```

instance variables

Data-Driven Design

File format.

```
% more 4body.txt
4          N
5.0e10     radius
                                 velocity       mass
-3.5e10 0.0e00 0.0e00  1.4e03 3.0e28
-1.0e10 0.0e00 0.0e00  1.4e04 3.0e28
 1.0e10 0.0e00 0.0e00 -1.4e04 3.0e28
 3.5e10 0.0e00 0.0e00 -1.4e03 3.0e28
    position
```

Constructor.

```
public Universe() {
   N = StdIn.readInt();
   radius = StdIn.readDouble();
   StdDraw.setXscale(-radius, +radius);
   StdDraw.setYscale(-radius, +radius);

   // read in the N bodies
   orbs = new Body[N];
   for (int i = 0; i < N; i++) {
      double rx   = StdIn.readDouble();
      double ry   = StdIn.readDouble();
      double vx   = StdIn.readDouble();
      double vy   = StdIn.readDouble();
      double mass = StdIn.readDouble();
      double[] position = { rx, ry };
      double[] velocity = { vx, vy };
      Vector r = new Vector(position);
      Vector v = new Vector(velocity);
      orbs[i] = new Body(r, v, mass);
   }
}
```

Principle of Superposition

Principle of superposition. Net gravitational force acting on a body is the sum of the individual forces.

```
// compute the forces
for (int i = 0; i < N; i++) {
   for (int j = 0; j < N; j++) {
      if (i != j) {
         f[i] = f[i].plus(orbs[j].forceTo(orbs[i]));
      }
   }
}
```

$$F_i = \sum_{i \neq j} \frac{G\, m_i\, m_j}{|r_i - r_j|^2}$$

Universe Data Type: Java Implementation

```
public class Universe {
   private final double radius;     // radius of universe
   private final int N;             // number of bodies
   private final Body[] orbs;       // array of N bodies

   public Universe() { /* see previous slide */ }

   public void increaseTime(double dt) {
      Vector[] f = new Vector[N];
      for (int i = 0; i < N; i++)
          f[i] = new Vector(new double[2]);
      for (int i = 0; i < N; i++)
          for (int j = 0; j < N; j++)
              if (i != j)
                  f[i] = f[i].plus(orbs[j].forceTo(orbs[i]));

      for (int i = 0; i < N; i++)
         orbs[i].move(f[i], dt);
   }

   public void draw() {
      for (int i = 0; i < N; i++)
         orbs[i].draw();
   }

   public static void main(String[] args) { /* see previous slide */ }

}
```

create universe

update the bodies

draw the bodies

simulate the universe

Odds and Ends

Accuracy. How small to make `dt`? How to avoid floating-point inaccuracies from accumulating?

Efficiency.

- Direct sum: takes time proportional to N^2

  $\Rightarrow$ not usable for large N.
- Appel / Barnes-Hut: takes time proportional to N log N time
- $\Rightarrow$ can simulate large universes.

3D universe. Use a 3D vector (only drawing code changes!).

Collisions.

- Model inelastic collisions.
- Use a softening parameter to avoid collisions.

$$F_i = \sum_{i \neq j} \frac{G\, m_i\, m_j}{|r_i - r_j|^2 + \varepsilon^2}$$

Extra Slides

N-Body Simulation

1. Setup initial distribution of particles.
   - Need accurate data and model of mass distribution.

2. Compute forces between particles.
   - Direct sum: N^2.
   - Appel / Barnes-Hut" N log N.

$$\mathbf{F}_i = \sum_{i \neq j} \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2}$$

ε = softening parameter
eliminates binary stars with r < ε
hard binaries can be important
source of energy

3. Evolve particles using ODE solver.
   - Leapfrog method balances efficiency and accuracy.
   - Truncation error = O(dt^2).
   - Symplectic.

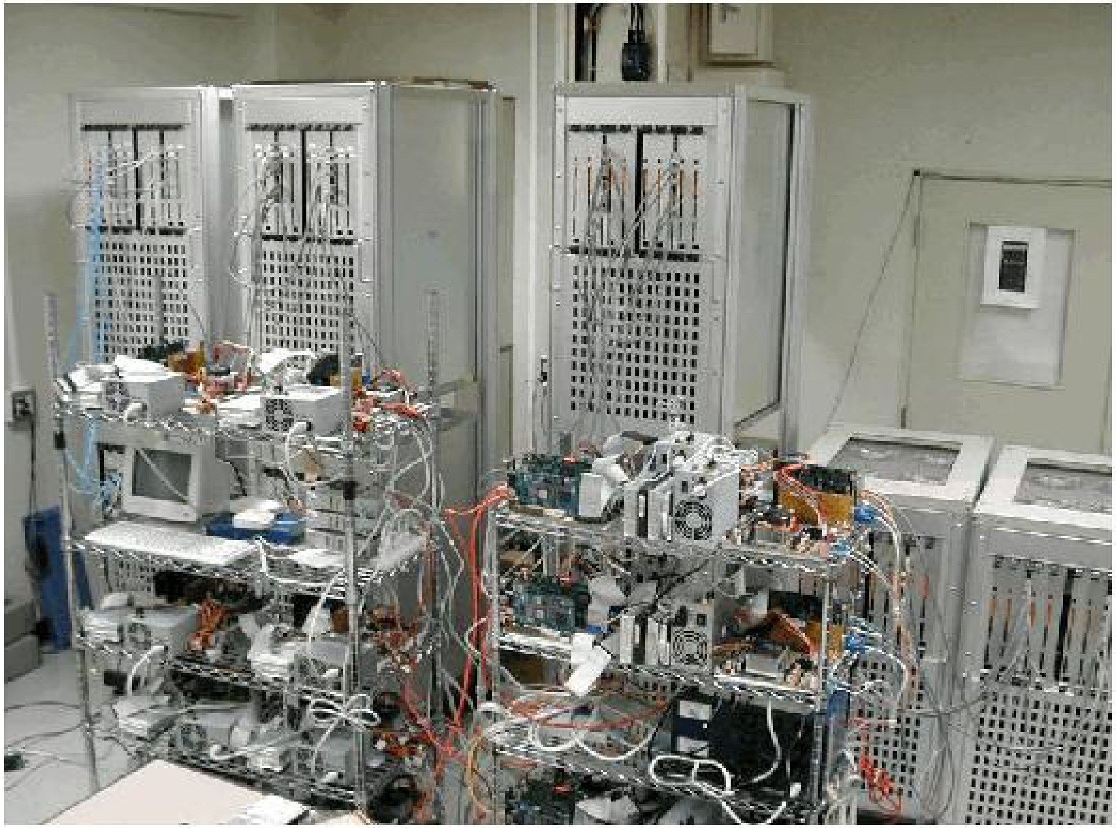
$$\frac{d\mathbf{X_i}}{dt} = \mathbf{V_i}$$

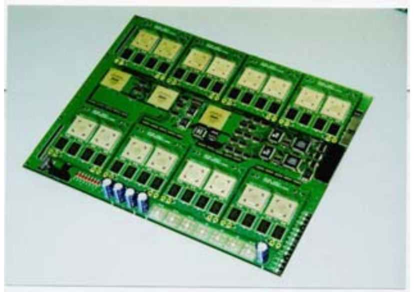
$$m_i \frac{d\mathbf{V_i}}{dt} = \mathbf{F_i}$$

4. Display and analyze results.

Solving the force problem with hardware.

GRAPE-6. Special purpose hardware to compute force.

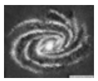




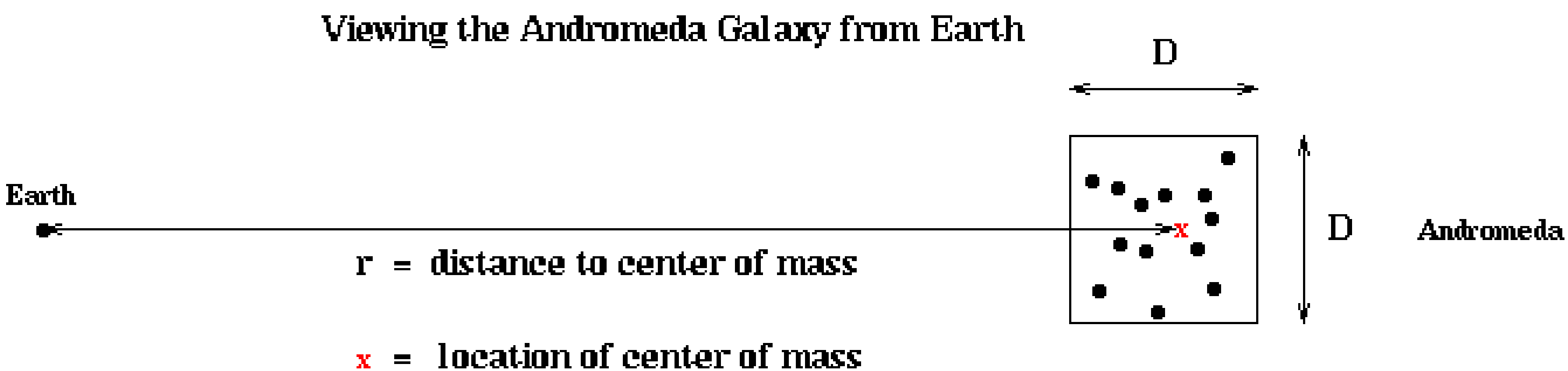
Jun Makino, U. Tokyo

Do we really need to compute force from every star for distant objects?

Solving the force problem with software -- tree codes



Distance = 25 times size



Organize particles into a tree. In Barnes-Hut algorithm, use a quadtree in 2D

A Complete Quadtree with 4 Levels